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Report on 'Invariant quantization and differential symmetries on AHS structures' by Dr. Josef Silhan.

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The Habilitation Thesis 'Invariant quantization and differential symmetries on AHS structures' consists of six articles authored, or co-authored by Silhan. The have all appeared in reputable international journals. In the more detailed description below I shall use the original numbering [13, 34, 20, 30, 23, 24] which the author refers to in the Preface to the Thesis.

The unifying theme is the theory of linear invariant operators, generalising the conformal Laplacian to manifolds with the AHS (almost Hermitian symmetric) structure. These structures correspond to a Cartan connection modelled on a compact Hermitian symmetric space G/P where G is a semi–simple Lie group, and P is its parabolic subgroup. The AHS structure are generalisations of differential geometry of a (pseudo) Riemannian manifolds. Two of the most interesting examples are

Conformal Geometry. An equivalence class of (pseudo) Riemannian metrics. Two metrics g and $\hat{g}$ are equivalent if $\hat{g} = \Omega^2 g$, where Ω is a smooth non-vanishing function. On a conformal manifold one can measure angles between vectors but not lengths.

Projective Geometry. An equivalence class of torsion-free connections. Two connections are equivalent if they share the same unparametrised geodesics. In this geometry there exists a path through any point in any direction but lengths and angles are not defined.

The local differential geometric structure is rich in both cases. One can define curvature as well as other invariant tensors and invariant differential operators. In his work Silhan proposes and develops a general theory of linear operators on AHS structures, ranging from symmetries and higher symmetries (relevant to integrable the separability theory of the Hamilton-Jacobi equations) to invariant quantisation (a procedure of determining an operator from its principal symbol).

Papers [13] and [34] are devoted to quantisation of principal symbols. The subject goes back to the early days of quantum theory. In 1925 Dirac proposed a construction (now known as the canonical quantisation) which replaces functions on the classical phase space by self-adjoint linear operators on a Hilbert space in a way which changes the Poisson brackets of functions to commutators of operators. It was subsequently realised (the so–called Groenewold-Van-Hove theorem - a coonsequence of the represention theory of $SL(2, R)$ on

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Hilbert spaces) that the canonical quantisation does not exist due to the operator ordering problem. The research has since then focused on weakening the quantisation axioms, as well as generalising the procedure to non–flat phase spaces.

In [13] Silhan, together with Andreas Cap developed a quantisation procedure of operators on general AHS spaces. The construction is very general, and applicable to both projective and conformal settings (both special cases of the AHS structures). The most important result of this paper is perhaps Theorem 5 (numbers as in the paper) giving a bound of the orders of operators corresponding to certain principal symbols. In [34] a theory of conformal quantisation has been developed by Silhan in a single–authored paper. The main result of this paper is the elegant Theorem 3.3 which gives an inductive formula for a quantisation of operators with non critical weights.

In [20] and [30] Silhan together with Rod Gover [20] and Jean-Philippe Michel and Fabian Radoux [30] generalised the seminal results of Michael Eastwood to describe symmetries of the higher powers of the Laplacian on conformally flat manifolds [20], and second order symmetries of a Laplacian on curved conformal manifolds [30]. In [20] the authors develop a notion of a generalised Killing tensor which in a certain sense (Theorems 2.1 and 2.4) characterised principal symbols of symmetries of the powers of Laplacians. In [30] the theory of second order symmetries of the conformal Laplacian is developed. The classification Theorems 4.8 and 4.11 are the key results of this work. I find the results of [30] especially interesting. Very little is known about higher order symmetries if the Weyl tensor doesn't vanish. There is certainly a need for a theory: some exact solutions to Einstein equations (e.g. the Kerr black hole) admit Killing tensors and conformal Killing tensors which enable the separability of the geodesic equations. The work of Silhan may eventually lead to a more systematic theory behind these isolated examples.

The papers [23] and [24] (written jointly with Hammerl, Somberg and Soucek) are devoted to the theory of prolongation linear operators, and more specifically [24] to the constructions of prolongation connections. Many problems in both projective and conformal differential geometries lead to linear systems of PDEs which are not closed: not all partial derivatives of the unknown functions are determined by the equations. A prolongation is a procedure of increasing the number of dependent variables (one regards the derivatives as 'new' functions) and cross–diferentiating aiming to determine all derivatives of the prolonged systems. This only works for a restricted class of equations (the so called systems of finite type - a seminal theory here was developed by Spencer and Kuranishi), but is applicable to many problems of current interest (e.g. metrisability of projective structures, existence of an Einstein or Kahler metric in a given conformal structure). The main result of [23] is the curved version of a prolongation of the BGG operators. This required the authors to construct a new normalisation condition for the tractor covariant derivatives (Theorem 1.2). The second paper [24] is focused on examples in projective and conformal geometries.

Conclusion. In my opinion Joseph Silhan is one of the emerging leaders in the field of parabolic geometries. On several occasions I have had a pleasure to attend the lectures he gave at international meetings - most recently in March this year, at the 1st International Conference in Differential Geometry in Fez. Dr Silhan is an excellent expositor. In his presentations he skillfuly separates the results which are of general interests, from the rather technical aspects of the formalism. This makes his lectures and contributions accessible to more general audiences of differential geometers - something which our subject benefits from!

The Thesis submitted by Dr. Josef Silhan, entitled 'Invariant quantization and differential symmetries on AHS structures' meets the requirements applicable to Habilitation theses in Geometry.

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