Masaryk University
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G-STRUCTURES, DIRAC OPERATORS WITH TORSION, AND SPECIAL SPINOR FIELDS

Habilitation Thesis

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Commentary

This thesis is dedicated to the study of G-structures, adapted connections (with torsion), and Dirac or twistor operators on Riemannian spin manifolds associated to such adapted connections, especially in the case where their torsion is a 3-form. We begin by briefly reviewing the state of the art in the field, and proceed by recalling results on the type of generalized Dirac and twistor operators that we are interested in and have motivated our research.

We consider the characteristic connection ∇^c and after assuming that its torsion 3-form T is ∇^c-parallel, we focus on the classification of special eigenspinors of such differential operators, as Killing spinors with torsion and twistor spinors with torsion. We show that in this case twistor spinors with torsion verify most of the structural properties of Riemannian twistor spinors. We also describe integrability conditions for such spinors, and under certain assumptions classify them in low dimensions 3, 6 and 7. Moreover, we provide a result on the relation of two eigenvalue estimates of generalized Dirac operators of this type, known by [**AF04a**, **A+13**]. Our results are also related to ∇^c-parallel spinors (such spinors represent the supersymmetries of admissible models in Type II string theory). In particular, we provide a new perspective on ∇^c-parallel spinors, based on a new spinorial formula encoding the action of the Ricci endomorphism on the spinor bundle in terms of the generalized Dirac operator. We then describe applications, as for example a new proof of the generalized Schrödinger-Lichnerowicz formula, under the assumption $\nabla^c T = 0$. In order to illustrate further applications we are based on Sasakian manifolds in dimension 5, nearly Kähler manifolds in dimension 6, and weak (or nearly parallel) G_2-manifolds in dimension 7.

Next we proceed with the classification of invariant affine or metric connections on compact, non-symmetric, effective, strongly isotropy irreducible homogeneous spaces G/K, where in terms of representation theory we compute the dimensions of the spaces of G-invariant affine and metric connections. For such manifolds we also describe the space of invariant metric connections with skew-torsion, proving that many of them admit connections with skew-torsion *not* induced by the Lie bracket family. This induces the classification of ∇-Einstein structures with skew-torsion which we also present. For the compact Lie group $\mathsf{U}(n)$ we classify all bi-invariant metric connections, and introduce a new family of bi-invariant connections with torsion of vectorial type.

We also focus on compact, simply connected, homogeneous 8-manifolds admitting invariant $\mathsf{Spin}(7)$-structures, and classify all canonical presentations G/H of such spaces, with G being simply connected. For each presentation we then exhibit explicit examples of invariant $\mathsf{Spin}(7)$-structures, describe their type according to their intrinsic torsion and analyze the associated $\mathsf{Spin}(7)$-connection with torsion, introduced by [**I04**].

The final part of this thesis is devoted to the intrinsic geometry of G-structures, where G is one of the Lie groups $\mathsf{SO}^*(2n)$, or $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$ $(n > 1)$. Here we provide the first systematic study of such geometric structures, a procedure that allows us to highlight them as the symplectic analogue of the well known almost hypercomplex/quaternionic Hermitian geometries (cf. [**AℓM96**]). This includes the decompositions of the corresponding intrinsic torsion modules into irreducible submodules, in terms of Spencer cohomology, and the description of minimal adapted connections with respect to certain normalization conditions. Hence we classify the algebraic types of such G-structures and study their first order (local) differential geometry. Besides other results, this includes the classification of those symmetric spaces admitting a torsion-free $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structure, and a local construction of torsion-free $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures with special symplectic holonomy, based on a previous result by Cahen-Schwachhöfer [**CS09**].

Applicant's contribution in terms of quality and content

The habilitation thesis at hand consists of five published articles:

A: ([**C16b**]) I. Chrysikos, *Killing and twistor spinors with torsion*, Annals of Global Analysis and Geometry, Vol. 49 (2016), 105–141.

B: ([**C17**]) I. Chrysikos, *A new $\frac{1}{2}$-Ricci type formula on the spinor bundle and applications*, Advances in Applied Clifford Algebras, Vol. 27, (4), (2017), 3097–3127.

C: ([**C+19**]) I. Chrysikos, C. O'Cadiz Gustad, H. Winther, *Invariant connections and ∇-Einstein structures on isotropy irreducible spaces*, Journal of Geometry and Physics, Vol. 138, (2019), 257–284.

D: ([**Aℓ+20**]) D. Alekseevsky, I. Chrysikos, A. Fino, A. Raffero, *Homogeneous 8-manifolds admitting invariant* $\mathrm{Spin}(7)$*-structures*, International Journal of Mathematics, Vol. 31, No. 8 (2020), 2050060 (33 pp).

E: ([**C+22a**]) I. Chrysikos, J. Gregorovič, H. Winther: *Differential geometry of* $\mathrm{SO}^*(2n)$*-type structures*, Annali di Matematica Pura ed Applicata (1923 -), 201, (2022) 2603–2662.

The applicant is sole author of [**C16b**] and [**C17**]. These articles are devoted to special spinor fields induced by non-integrable G-structures on Riemannian spin manifolds, and other applications from the theory of metric connections with skew-torsion and generalized Dirac or Penrose operators. Hence we need to mention that they establish a source of motivation for the research topics in [**C+19**, **Aℓ+20**], as well.

The article [**C+19**] is a joint work with C. O'Cadiz Gustad and H. Winther. Here the goal was to extend some results from [**C16a**], the latter representing the very first work of the applicant in the area of metric connections with skew-torsion. In Section 2 of [**C+19**] the results are mainly obtained by the applicant. In the rest Sections 3, 4, and 5 most of the results can be viewed as the product of the collaboration of all three authors, hence it is not possible to assign the credit for the results here just to one particular author. The computer computations in Section 4 were mainly done by C. O'Cadiz Gustad, by extending the code of LiE package (http://www-math.univ-poitiers.fr/ maavl/LiE/), with many parallel contributions of both the applicant and H. Winther. During this work the authors used Git (https://git-scm.com) to keep track and update the files. All the authors in this article have the same amount of writing, reviewing or editing the final draft, and applicant's work is roughly 33%.

The article [**Aℓ+20**] is a joint work with D. Alekseevsky, A. Fino and A. Raffero. It was initiated the second year of the period that the applicant was working in the Department of Mathematics "Giuseppe Peano", Università degli Studi di Torino in Turin, as an owner of an INdAM grant co-funded by Marie Curie Actions. It was essentially completed during 2019, a period that the applicant was employed in the University of Hradec Králové, and published in 2020. Our goal in this work was to focus on the Lie group $\mathsf{Spin}(7)$ and obtain the analogue of the known classification of invariant G_2-structures on homogeneous 7-manifolds ([**R10**, **LM12**]), but also highlight the adapted invariant connection with skew-torsion (cf. [**I04**]). The applicant signs for writing the first manuscript of the proof of Theorem A in [**Aℓ+20**], and in particular of the results presented in Section 5, which have a topological character. The presentation of these results, together with the preliminary results in Section 2, 3 and 4, were improved in the final draft by all the four authors. The applicant also signs for the proof of Theorem B, and in particular for most of the results on the infinite family $C_{k,\ell,m}$ and the Calabi–Eckmann manifold $(\mathsf{SU}(3)/\mathsf{SU}(2)) \times \mathsf{SU}(2)$, presented from the beginning of Section 6 till Corollary 7.13 in Section 7.2, page 2050060-27. Finally, the applicant signs for the results presented in Appendix A, and Appendix B. All the authors in [**Aℓ+20**] have the same amount of reviewing or editing the final draft, and applicant's contribution to this article is roughly 40%.

The article [**C+22a**] is a joint work with J. Gregorovič and H. Winther. It was initiated and completed during the period that the applicant was the leader of a Junior Research Team in the Faculty of Science, University of Hradec Králové, supported by Czech Science Foundation. The main motivation here was to establish the symplectic analogue of almost hypercomplex-Hermitian and almost quaternion-Hermitian geometries, and derive the intrinsic torsion modules of the corresponding G-structures, as an analogue of the work done in [**AℓM96**]. This work emerged from discussions of the applicant with his collaborator H. Winther. Later on, J. Gregorovič joined the project as a member of the applicant's research team, and an expert in similar topics (cf. [**G13**]). The article [**C+22a**] is the product of a strong collaboration between all the three authors, and hence it is not possible to assign the credit for most of the results included in [**C+22a**] just to one particular author. It contains the first results of a lengthy joint effort, which was continued with the results presented in [**C+22b**]. During the writing of [**C+22a**] the applicant had concentrated more on Sections 3, 4 and 5. The biggest part of this work was completed during the Covid-19 pandemic, so the authors used Git to keep track/update the files and had established a weekly online meeting. All the authors in this article have the same amount of writing, reviewing or editing the final draft, and a quantitative contribution of applicant's work is roughly 33%.

Bibliographic record of the publications submitted to this thesis

The public version of this thesis includes only the text of the Article C. In particular, the full texts of the publications A, B, D, and E are not attached in the public version, due to licence restrictions.

Acknowledgements

I would like to acknowledge and give my warmest thanks to all my colleagues from Masaryk University, from the University of Hradec Králové and from abroad, especially to my colleagues from Institute of Information Security in Moscow, University of Torino and University of Hamburg. Special thanks go to my family for their continuous support.

Contents

1 Summary

1.1 Introduction

The theory of G-*structure*s, or equivalently of *linear connections*, is a fast-moving field with many new discoveries both in differential geometry and in theoretical or mathematical physics. The equivalence between these two notions relies on the *holonomy* of any such connection, which is a central tool in differential geometry having a series of important applications (one may think for example of the so called "holonomy principle"). In the Riemannian context, there is a tremendous research on the geometry and topology of manifolds with *special holonomy*. Especially manifolds admitting ∇^g-parallel spinors and hence Ricci-flat holonomy reductions (as Calabi-Yau manifolds, parallel G_2- and parallel $\mathsf{Spin}(7)$-manifolds) have a key role in string theory compactifications and M-theory, where they support vacuum solutions (for more details on such manifolds and their applications we refer to [**J00**, **D03**, **BG08**]).

First order G-structures can be successfully treated in terms of their *intrinsic torsion*, and non-trivial (intrinsic) torsion is synonymous to non-integrable geometries. This notion became more famous during 80s and 90s, after the lengthy papers of Guillemin and Sternberg on Spencer cohomology. Good examples involve the works of Gray and his collaborators on almost-Hermitian structures and G_2-structures ([**GH80**, **FG82**]), and the works of Salamon on almost quaternionic geometries [**S82**, **S86**] (see also [**F86**] for $\mathsf{Spin}(7)$-structures, [**CG90**, **F95**] for almost contract metric geometries, and [**AℓM96**] for quaternionic-like geometries). Nowadays, intrinsic torsion has become a folklore topic, and it is known for many different kinds of G-structures ([**Cℓ01**, **N08**, **AFH13**]). Intrinsic torsion is also adapted to the more general framework of parabolic geometries where often more algebraic tools are available (e.g., Kostant's theorem on Lie algebra cohomology [**K61**]), and it is possible to provide more general constructions highlighting the theory of differential complexes and BGG sequences ([**ČSS**]). For example, the intrinsic torsion of parabolic almost conformally symplectic structures is described in [**ČS17a**, **ČS17b**], where are also constructed differential complexes intrinsically associated to special symplectic connections.

To summarize, it is fair to say that the field of both integrable and non-integrable geometries is much deeper the days written this thesis, with many new exciting developments, but there are still many interesting things to discover and learn (for the interested reader we cite the articles [**F+18**] for new discoveries related to the realization of exceptional holonomies and [**CV15**, **FH18**] for the construction of the first known examples of non-homogeneous nearly Kähler structures).

Notice that the presence of torsion characterizes non-integrable geometries. Thus, in the Riemannian case (where one is interested in G-structures with G being a closed subgroup of $\mathsf{SO}(n)$), the Levi-Civita connection is replaced by a metric connection with torsion preserving the corresponding G-structure. When such a connection exists and has totally antisymmetric torsion (in short, skew-torsion), it is referred to as the *characteristic connection*, denoted here by ∇^c. It is known that for all the cases where ∇^c exists, it is unique, while its (characteristic) torsion form T can been computed explicitly in terms of the underlying geometric data (see [FI02]). Adapted connections with skew-torsion and their holonomy theory provide a tool of studying non-integrable Riemannian geometries and their special spinor fields, an approach that often offers unexpected new aspects even of well-acquainted objects (as for example tools for classifying naturally reductive spaces, see [A+15]). Very recently, this approach has been extended to the Lorentzian case, as well (see [IG22]).

1.2 Motivation

In the Riemannian context non-integrable geometries become important for many reasons. Firstly, there are classes of such geometries that induce Einstein metrics and satisfy some nice topological properties (such examples are nearly Kähler manifolds, weak G_2-manifolds or 3-Sasakian manifolds, which are all spin and Einstein). It is by now known that these Einstein metrics are related to the existence of Killing spinors ([FK89, FK90, Gr90, BG08]). Recall that Killing spinors realize the equality case of the estimate of the first eigenvalue of the Riemannian Dirac operator on closed Riemannian spin manifolds with positive scalar curvature, a celebrated result by Friedrich [F80]. Later, Lichnerowicz enforced the role of such spinors, by establishing a link to twistor theory by proving that in the compact case the space of twistor spinors coincides, up to a conformal change of the metric, with the space of Killing spinors ([L87]). Let us finally mention that real Killing spinors can be classified in terms of Bär's cone construction ([B93]), and as one can expect they have a key role is many supergravities theories, where solutions are based on manifolds carrying such spinors. For example, in [Aℓ+19] we construct bosonic models based on products of 7-dimensional weak G_2-manifolds and 4-dimensional Lorentian Einstein spaces, generalizing an older construction by [B+02] (for more details on 11-dimesnional supergravity we refer to [D03, F13, Aℓ+19, CG20, C+23a] and the references therein).

Another reason why the non-integrable geometries are important is the fact that many of them support ∇^c-parallel spinors, a situation that applies to the construction of admissible solutions in string theory. Indeed, in the early 80's physicists tried to incorporate torsion into superstring theories in order to get a physically flexible model. In Type II string theory for example, the equations in the string frame can be written in the following way (see [Str86] and see also [IP01, FI02, A03, I04, II05, A06])

$$\nabla\psi = 0\,, \qquad (d\,\Phi - \frac{1}{2}T)\cdot\psi = 0$$

where Φ is a scalar function called the *dilation*, T is a 3-form (representing the field strength), ψ is a a spinor field, "$\cdot$" is the Clifford multiplication, and ∇ is the lift onto the spinor bundle of a metric connection with totally skew-symmetric torsion T. The number of preserving supersymmetries is determined by the number of solutions of these equations, hence ∇-parallel spinors are immediately involved in the theory and are of great importance.

Part of this thesis is based on the articles [C16b, C17], which are about metric connections with *parallel skew-torsion* preserving non-integrable G-structures on Riemannian manifolds, with a focus on the spin case in terms of the induced Dirac and Penrose operators and their special eigenspinors (as *Killing spinors with torsion and twistor spinors with torsion*, see below). The results from [C17] are also related with ∇^c-parallel spinors, providing new integrability conditions, and hence they have potential applications in string theory, as it was briefly explained above.

Parallel skew-torsion is an extra condition and manifolds endowed with such connections provide a natural generalization of naturally reductive spaces. There are many known examples, as nearly Kähler manifolds, weak G_2-manifolds, Sasakian manifolds and other (see [A06]). On the other hand, Dirac operators on Riemannian spin manifolds provide a powerful tool for the treatment of various problems in geometry, analysis and theoretical physics ([B+91, F00]). The investigation of adapted Dirac operators on non-integrable Riemannian structures, and hence *Dirac operators with torsion*, was initiated by the work of Bismut on Hermitian manifolds and the local index theorem ([B89]). A decade later Kostant [K99] introduced the so called *cubic Dirac operator*, a purely algebraic object that can be interpreted as the Dirac operator induced by an invariant connection with skew-torsion on a naturally reductive space, as it was shown shortly after by Agricola in [A03]. It is also known that these operators and their related theory can be successfully carried out in a non-homogeneous setting, see [DI01, FI02, AF04a].

There are many reasons why such spinorial operators are important. For example, in the homogeneous setting Dirac operators with torsion provide a generalization of the Dirac operator of Parthasarathy ([P72]) on symmetric spaces, an algebraic object whose square can be successfully studied in terms of Casimir operators and representation theory. This situation appropriately extends to the homogeneous cubic Dirac operator and one can present estimates for the first eigenvalue of this operator in representation theoretical terms ([K99, A03]).

Such eigenvalue estimates can be presented in the non-homogeneous case as well, see [AF04a, AF04b, A+13]. Moreover, one can relate the equality case in one of these estimates with *twistor spinors with torsion* (TsT) and in some special cases *Killing spinors with torsion* (KsT). Such spinors were introduced in [A+13] and are natural generalizations of the corresponding notions known from the Riemannian setting to Riemannian spin manifolds admitting metric connections with skew-torsion. One of the results in the mentioned article is that for 6-dimensional nearly Kähler structures such spinors realize the equality case for eigenvalue estimates of the cubic Dirac operator

$$\not{D} = D^g + \frac{1}{4}T$$

(we may think of $\not{D}$ as the Dirac operator associated to the connection with torsion $T/3$). In [C16b] we prove that this statement makes sense also in dimension 7 for nearly parallel G_2-manifolds, see below for more details on these estimates, which both occur in the presence of parallel torsion, $\nabla^c T = 0$.

In [C17] we present a new approach attacking Dirac or Penrose operators induced by metric connections with skew-torsion, and moreover a new approach for studying ∇^c-parallel spinors, that is, spinors satisfying the equation

$$\nabla^c \psi = 0\,,$$

and more generally $\nabla^s\psi = 0$, where ∇^s denotes the lift of the 1-parameter family of metric connections with torsion $4sT$ ($s \in \mathbb{R}$), under the condition $\nabla^c T = 0$, where T is the torsion 3-form of ∇^c. On the other hand, under the same assumption, in [**C16b**] we study Killing spinors with torsion, and twistor spinors with torsion. We derive integrability conditions for their existence, and under certain assumptions present a bijective correspondence with ∇^c-parallel spinors, a fact that allows us to classify such special KsT or TsT in low dimensions 3, 6 and 7.

Notice that by now it is well known that non-symmetric homogeneous (reductive) spaces provide a great source of manifolds where many non-integrable structures can be explicitly described. Motivated by this fact, in [**Aℓ+20**] we obtain the classification of compact, simply-connected homogeneous 8-manifolds admitting invariant $\mathsf{Spin}(7)$-structures. Recall that manifolds admitting a torsion-free $\mathsf{Spin}(7)$-structure admit ∇^g-parallel spinors and hence are Ricci flat ([**B66**]), in particular they appear in Berger's list of irreducible Riemannian holonomies. On the other hand, non-integrable $\mathsf{Spin}(7)$-structures play an important role in theoretical physics since they admit parallel spinors with respect to a metric connection with skew-torsion, see for example [**I04**, **II05**]. In [**Aℓ+20**] we combine topology with results by [**K88**] to obtain the classification mentioned above. Then, we specify for any coset in our list an invariant $\mathsf{Spin}(7)$-structure induced by an invariant 4-form, and study its geometric features. This allows us to describe the algebraic type of such $\mathsf{Spin}(7)$-structures, in terms of [**F86**], and also analyse the associated $\mathsf{Spin}(7)$-connection with skew-torsion (see [**I04**]).

The article [**C+19**] is also devoted to the homogenous case. In this article we present the classification of invariant affine and metric connections and of their torsion classes (in terms of Cartan [**C25**], see also [**AF04a**]), on all compact non-symmetric, strongly isotropy irreducible homogenous spaces G/K, where G is assumed to act effectively. Our approach combines the classical theory of Wang on invariant connections [**W58**], with representation theory of semisimple Lie algebras. Recall that a connected effective homogeneous space G/K is called isotropy irreducible if K acts irreducibly on $T_o(G/K)$ via the isotropy representation. If the identity component K_0 of K also acts irreducibly on $T_o(G/K)$, then G/K is called *strongly isotropy irreducible* (SII for short). Obviously, any strongly isotropy irreducible space is also isotropy irreducible but the converse is in general false. Non-symmetric strongly isotropy irreducible homogeneous spaces were originally classified by Manturov (see for example [**WZ93**]) and were later studied by Wolf [**W84**] and others. A conceptual relationship between symmetric spaces and SII spaces was explained in [**WZ93**]. More recently, isotropy irreducible homogeneous spaces endowed with their canonical connection ∇^c was shown to have a special relationship with geometric structures with torsion (see [**CℓS04**, **Cℓ01**]).

In [**C+19**] we also specify the dimension of the space of invariant metric connections with skew-torsion on compact non-symmetric SII spaces, hence our paper provides many new examples of homogeneous manifolds admitting invariant metric connections with skew-torsion, *not* all of them induced by the Lie bracket family. This last step allows us to classify all existent invariant ∇-Einstein structures with skew-torsion, as well. The notion of ∇-Einstein structures was introduced in [**AF14**] and though it does not provide a generalization of Einstein's field equations, there are many well known examples of ∇^c-

Einstein manifolds (as nearly Kähler manifolds, weak G_2-manifolds and other). Hence it is reasonable to exam this notion separately. Such geometric structures consist of a n-dimensional connected Riemannian manifold (M, g) endowed with a metric connection ∇ which has non-trivial skew-torsion T and whose Ricci tensor has symmetric part a multiple of the metric tensor, that is, (see also [**FI02**, **AF14**, **C16a**, **C16b**])

$$\mathsf{Ric}_S^{\nabla} = \frac{\mathsf{Scal}^{\nabla}}{n} g.$$

By [**AF14**] it is known that ∇-Einstein structures can be characterized variationally (see also [**C19**] for a related result), and recently there is some interest in the classification of such structures, especially in a homogeneous setting where representation theory and other tools are available, see for example [**C16a**, **D+16**].

The final paper submitted in this thesis ([**C+22a**]) is *not* related to skew-torsion, as we introduced it above, but it relies on the more general notion of intrinsic torsion. This article is the first in a series of articles (see [**C+22b**, **C+23b**]) devoted to the (local) differential geometry of a new type of G-structures on quaternionic-like manifolds, corresponding to the Lie groups $\mathsf{SO}^*(2n)$ and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$, respectively. As we will explain below, such G-structures do not support a Riemannian metric, but they preserve a "compatible" almost symplectic form ω. Hence, they form the symplectic analogue of the better understood almost hypercomplex-Hermitian structures and almost quaternionic-Hermitian structures, respectively (see [**S86**, **S91**, **AℓM96**] for details on these geometries).

At this point we should mention that the Lie group $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$ belongs to the list of non-metric *exotic holonomies*, in particular it is related with the notion of *special symplectic holonomy*, introduced by Cahen–Schwachöffer in [**CS09**]. On the other hand, it is known by Bryant ([**Br96**]) that torsion-free affine connections with (irreducible) full holonomy group $\mathsf{SO}^*(2n)$ cannot exist, since Berger's first criterion fails for the Lie algebra of $\mathsf{SO}^*(2n)$. Our approach in [**C+22a**] differs from those which are mainly devoted to the torsion-free case ([**Br96**, **S01a**, **S01b**, **CS09**]) and in particular a main contribution of this work is based on the establishment of the local geometry of $\mathsf{SO}^*(2n)$- or $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures, via a geometric approach based on defining tensor fields, adapted frames, adapted connections, intrinsic torsion modules, minimal connections and normalization conditions, and finally examples (first order integrability conditions are presented in [**C+22b**], while the curvature is studied in [**C+23b**]). In this way we proceed systematically with a differential-geometric treatment of manifolds carrying such structures and highlight some parts of their unknown till now intrinsic geometry, both in the integrable and non-integrable case.

1.3 Overview of results

(A) Killing and twistor spinors with torsion. Consider a connected Riemannian spin manifold (M^n, g) $(n \geq 3)$ endowed with a 1-parameter family of metric connections with (totally) skew-symmetric torsion, say $4sT$ for some non-trivial 3-form T on M and $s \in \mathbb{R}$. This has the form

$$\nabla^s = \nabla^g + 2sT$$

and joins the Levi-Civita connection ($s = 0$) with the connection with torsion T ($s = 1/4$), that is, $\nabla^{1/4} = \nabla^g + \frac{1}{2}T$, which by some abuse of notation we will denote by ∇^c. We use the same symbol for the lift of ∇^s on the spinor bundle $\Sigma \to M$ over M, given by

$$\nabla^s_X\varphi = \nabla^g_X\varphi + s(X \lrcorner T)\cdot\varphi$$

for any $X \in \Gamma(TM)$ and $\varphi \in \Gamma(\Sigma)$.[1] After identifying $TM \cong T^*M$ via the metric tensor, the associated Dirac operator is defined by the composition

$$D^s = \mu \circ \nabla^s : \Gamma(\Sigma) \xrightarrow{\nabla^s} \Gamma(TM \otimes \Sigma) \xrightarrow{\mu} \Gamma(\Sigma)\,.$$

Locally, it has the form

$$D^s(\varphi) = \sum_i e_i \cdot \nabla^s_{e_i}\varphi = \sum_i e_i \cdot (\nabla^g_{e_i}\varphi + s(e_i \lrcorner T)\cdot\varphi) = D^g(\varphi) + 3sT\cdot\varphi\,,$$

where $D^g \equiv D^0 := \mu \circ \nabla^0$ is the Riemannian Dirac operator. Notice that

$$D^s = \begin{cases} D^0 = D^g, & \text{for } s = 0;\\ D^{1/4} = D^c = D^g + \frac{3}{4}T, & \text{for } s = 1/4;\\ D^{1/12} = \not{D} = D^g + \frac{1}{4}T, & \text{for } s = 1/12. \end{cases}$$

Let us assume from now that $\nabla^c T = 0$, and consider the cubic Dirac operator $\not{D} = D^g + \frac{1}{4}T$. In this case for the square of $\not{D}$ the following formula of Schrödinger-Lichnerowicz type is known (apply [**A+13**, Thm. 2.2] for $s = 1/4$ and see also [**DI01**, **AF04a**, **AF04b**])

$$\not{D}^2 = \Delta_T + \frac{1}{4}\,\mathsf{Scal}^g - \frac{1}{4}T^2 + \frac{1}{8}\|T\|^2\,,$$

where $\Delta_T := (\nabla^c)^*\nabla^c = -\sum_i \nabla^c_{e_i}\nabla^c_{e_i} + \nabla^c_{\nabla^g_{e_i}e_i}$ denotes the spinor Laplacian associated to the connection ∇^c. Because $\nabla^c T = 0$, the square $\not{D}^2$ commutes with the symmetric endomorphism T. In particular, T acts on spinors with real constant eigenvalues [**A+05**, Thm. 1.1] and the spinor bundle decomposes into a direct sum of T-eigenbundles preserved by ∇^c:

$$\Sigma = \bigoplus_{\gamma\in\mathrm{Spec}(T)} \Sigma_\gamma\,, \quad \text{with} \quad \nabla^c\Sigma_\gamma \subset \Sigma_\gamma\,, \quad \forall\, \gamma \in \mathrm{Spec}(T)\,.$$

Similarly, the space of sections decomposes as $\Gamma(\Sigma) = \bigoplus_{\gamma\in\mathrm{Spec}(T)}\Gamma(\Sigma_\gamma)$. Then, from the generalized Schrödinger-Lichnerowicz formula and for the smallest eigenvalue λ_1 of the square $\not{D}^2$ restricted on Σ_γ it follows that (see [**AF04a**, **A+13**])

$$\lambda_1(\not{D}^2|_{\Sigma_\gamma}) \geq \frac{1}{4}\,\mathsf{Scal}^g_{\min} + \frac{1}{8}\|T\|^2 - \frac{1}{4}\gamma^2 := \beta_{\mathrm{univ}}(\gamma)\,.$$

The equality occurs if and only if Scal^g is constant and φ is ∇^c-parallel, i.e.,

$$\nabla^g_X\varphi + \frac{1}{4}(X \lrcorner T)\cdot\varphi = 0\,.$$

[1]We denote by $X\cdot\varphi := \mu(X\otimes\varphi)$ the Clifford multiplication $\mu : TM\otimes\Sigma \to \Sigma$ at the bundle level, a notation that naturally extends to p-forms.

These are the spinors fields that we are mainly interested in this thesis. A very similar expression to this estimate holds on the whole spinor bundle $\Sigma = \oplus_\gamma \Sigma_\gamma$, where one has to consider the maximum of the possible different eigenvalues $\{\gamma_1^2, \ldots, \gamma_q^2\}$.

Let us also consider the Penrose or twistor operator P^s associated to ∇^s, that is, the differential operator defined by the composition

$$\Gamma(\Sigma) \xrightarrow{\nabla^s} \Gamma(TM \otimes \Sigma) \xrightarrow{p} \Gamma(\ker \mu), \quad P^s = p \circ \nabla^s,$$

where $p : TM \otimes \Sigma \to \ker \mu \subset TM \otimes \Sigma$ is the orthogonal projection onto the kernel of the Clifford multiplication. Locally one has

$$p(X \otimes \varphi) := X \otimes \varphi + \frac{1}{n} \sum_{i=1}^{n} e_i \otimes e_i \cdot X \cdot \varphi \,, \quad P^s \varphi := \sum_{i=1}^{n} e_i \otimes \{\nabla^s_{e_i} \varphi + \frac{1}{n} e_i \cdot D^s \varphi\} \,.$$

Definition 1.1. (a) A Killing spinor with torsion (KsT) is a non-trivial spinor field $\varphi \in \Gamma(\Sigma)$ satisfying the relation

$$\nabla^s_X \varphi = \zeta X \cdot \varphi, \quad (*)$$

for any vector field X on M, and some $s \neq 0$ and $\zeta \neq 0$ (ζ is referred to as the Killing number).

(b) A *twistor spinor with torsion* (TsT in short), is a section $\varphi \in \Gamma(\Sigma)$ belonging to the kernel of the Penrose operator, that is, satisfying the twistor equation with respect to ∇^s for some $s \neq 0$:

$$\nabla^s_X \varphi + \frac{1}{n} X \cdot D^s \varphi = 0 \,. \quad (**)$$

For the first eigenvalue of $\not{D}^2$ restricted on Σ_γ there is a second estimate, the so-called *twistorial estimate* introduced in [A+13]. This is defined by

$$\lambda_1(\not{D}^2|_{\Sigma_\gamma}) \geq \frac{n}{4(n-1)} \mathsf{Scal}^g_{\min} + \frac{n(n-5)}{8(n-3)^2} \|T\|^2 + \frac{n(4-n)}{4(n-3)^2} \gamma^2 := \beta_{\mathrm{tw}}(\gamma) \,,$$

and the equality appears if and only if φ is a twistor spinor with torsion for the parameter $s = (n-1)/4(n-3)$, and moreover $\mathsf{Scal}^g =$ constant. A similar expression to the one given above holds for the whole spinor bundle.

In the presence of a ∇^c-parallel spinor $0 \neq \varphi \in \Sigma_\gamma$, the inequality $\beta_{\mathrm{tw}}(\gamma) \leq \beta_{\mathrm{univ}}(\gamma)$ needs to hold. For $n \leq 8$, this fact implies the inequalities [A+13, Lem. 4.1]

$$0 \leq 2n\|T\|^2 + (n-9)\gamma^2 \,, \qquad \mathsf{Scal}^g \leq \frac{9(n-1)}{2(9-n)} \|T\|^2 \,.$$

Here, the equality takes place if and only if the universal estimate coincides with the twistorial estimate. It is an interesting question to check if these twistor spinors with torsion are also some kind of Killing spinors and what the geometric inclusions are when the two estimates coincide, if any. This was part of our motivation for the results presented below.

Our article [C16b] begins with the investigation of the geometry of manifolds (M^n, g, T) admitting non-trivial twistors spinors with torsion with respect to the family ∇^s, under the assumption $\nabla^c T = 0$. We first computed the Ricci tensor and scalar curvature on such manifolds.

Theorem 1.2. *When $\nabla^c T = 0$, any non-trivial twistor spinor $\varphi \in \mathsf{Ker}(P^s)$ satisfies the following relations*

$$-\frac{1}{2}\,\mathsf{Ric}^s(X)\cdot\varphi = -\frac{8s}{n}(X\lrcorner T)\cdot D^s(\varphi) + \frac{n-2}{n}\nabla^s_X\big(D^s(\varphi)\big) - \frac{1}{n}X\cdot(D^s)^2(\varphi) - s(3-4s)(X\lrcorner\sigma_T)\cdot\varphi\,,$$

$$\frac{1}{2}\,\mathsf{Scal}^s\,\varphi = -\frac{24s}{n}T\cdot D^s(\varphi) + \frac{2(n-1)}{n}(D^s)^2(\varphi) - 4s(3-4s)\sigma_T\cdot\varphi\,,$$

for all $X \in \Gamma(TM)$.

Notice for $s = 0$, i.e., for the Riemannian connection (zero torsion $T \equiv 0$), this theorem reduces to a basic result of Lichnerowicz [**L87**] about Riemannian twistor spinors (see also [**B+91**, pp. 23-24], or [**F00**, pp. 122-123]):

$$\begin{aligned}\nabla^g_X\big(D^g(\varphi)\big) &= \frac{n}{2(n-2)}\Big[-\mathsf{Ric}^g(X)\cdot\varphi + \frac{\mathsf{Scal}^g}{2(n-1)}X\cdot\varphi\Big] = \frac{n}{2}\,\mathrm{Sch}^g(X)\cdot\varphi,\\ (D^g)^2(\varphi) &= \frac{n\,\mathsf{Scal}^g}{4(n-1)}\varphi,\end{aligned}$$

where $\mathrm{Sch}^g(X) := \frac{1}{n-2}\big[-\mathsf{Ric}^g(X)\cdot\varphi + \frac{\mathsf{Scal}^g}{2(n-1)}X\big]$ is the endomorphism induced by the Schouten tensor of ∇^g. In [**C16b**] we proved that via the Theorem 1.2 one can provide a generalization of these formulas (that is, for non-trivial elements in $\mathsf{Ker}(P^s)$).

The most important consequences of the twistorial $\frac{1}{2}$-Ric^s-formula described above can be encoded as follows (see Corollary 2.6 and Proposition 2.7 in [**C16b**]).

Corollary 1.3. *Let (M^n, g, T) $(n \geq 3)$ be a connected Riemannian spin manifold with $\nabla^c T = 0$. Then,*

a) The kernel of the twistor operator is a finite dimensional space, and $\dim_{\mathbb{C}}\mathsf{Ker}(P^s) \leq 2^{[\frac{n}{2}]+1}$.

b) If φ and $D^s(\varphi)$ vanish at some point $p \in M$ and $\varphi \in \mathsf{Ker}(P^s)$, then $\varphi \equiv 0$.

c) Any zero point of a twistor spinor with torsion $0 \neq \varphi \in \mathsf{Ker}(P^s)$ is isolated, i.e., the zero-set of φ is discrete.

These results show that, under the condition $\nabla^c T = 0$, twistor spinors with torsion (TsT) satisfy the same structural properties as the Riemannian twistor spinors. Notice however that in the Riemannian case the space $\mathsf{Ker}(P^g)$ is in addition a conformal invariant of (M^n, g), which is *not* in general the case in the presence of torsion (see however [**DI01**] for a remark in dimension 4). On the other hand, applying the above theorem on Killing spinors with torsion (KsT), we obtain integrability conditions for this kind of spinors, see [**C16b**, Corol. 2.3] (also known from [**B-B12**, Lem. 1.14], proved however in a different way, see also [**A+13**]).

Next we focus on KsT or TsT which are the same time parallel with respect to the characteristic connection ∇^c. Although this is a strong condition, we successfully develop a theory handling the situation, which is illustrated via examples by using for instance nearly Kähler manifolds in dimension 6 and weak G_2-manifolds in dimension 7.

The first step in our approach is the following theorem, which provides a criterium to decide when a ∇^c-parallel spinor is a real Killing spinor.

Theorem 1.4. *Let (M^n, g, T) be a compact connected Riemannian spin manifold with $\nabla^c T = 0$ and positive scalar curvature, carrying a non-trivial spinor field $\varphi_0 \in \Gamma(\Sigma)$ such that*

$$\nabla^c_X \varphi_0 = \nabla^g_X \varphi_0 + \frac{1}{4}(X \lrcorner T)\cdot\varphi_0 = 0, \quad \forall\, X \in \Gamma(TM), \quad T\cdot\varphi_0 = \gamma\varphi_0, \quad \text{for some } \gamma \in \mathbb{R}\backslash\{0\}. \tag{1.1}$$

Then, φ_0 is a real Killing spinor (with respect to g) if and only if the (constant) eigenvalue $0 \neq \gamma \in \mathrm{Spec}(T)$ satisfies the equation

$$\gamma^2 = \frac{4n}{9(n-1)}\,\mathsf{Scal}^g\,. \quad (\dagger)$$

If this is the case, then the Killing number is given by $\kappa := 3\gamma/4n$ and the following holds

$$(X \lrcorner T)\cdot\varphi_0 + \frac{3\gamma}{n}X\cdot\varphi_0 = 0\,, \quad \forall\, X \in \Gamma(TM)\,.$$

For $n \leq 8$, the condition $(\dagger)$ is equivalent to

$$\gamma^2 = \frac{2n}{9-n}\|T\|^2\,, \quad \textit{or} \quad \mathsf{Scal}^g = \frac{9(n-1)}{2(9-n)}\|T\|^2$$

and if this is the case, then the action of the symmetric endomorphism dT on φ_0 is given by $dT\cdot\varphi_0 = -\frac{3\gamma^2(n-3)}{2n}\varphi_0$.

The proof of this result is essentially based on some of the geometric conclusions imposed by a ∇^c-parallel spinor, in combination with a fundamental result on Killing spinors (cf. [**B+91**] pp. 20, 31, and see also the proof of Proposition 3.2 in [**C16b**]). Examples of structures satisfying this result are nearly Kähler structures, weak G_2-structures, and also the Killing spinors in S^3, see below.

Example 1.5. Let (M^6, g, J) be a (strict) 6-dimensional nearly Kähler manifold. Such a manifold admits two ∇^c-parallel spinors $\varphi^{\pm}$ lying in Σ_γ with $\gamma = \pm 2\|T\|$ ([**FI02**]). The scalar curvature is given by $\mathsf{Scal}^g = \frac{15}{2}\|T\|^2$ and this coincides with the quantity $\frac{9(n-1)}{4n}\gamma^2 = \frac{9(n-1)}{2(9-n)}\|T\|^2$. Therefore, by the above theorem the spinors $\varphi^{\pm}$ should be real Killing spinors with Killing number

$$\kappa = 3\gamma/4n = \pm\|T\|/4\,,$$

which is a well-known result by [**Gr90**]. Moreover, our theorem says that $dT\cdot\varphi^{\pm} = -\frac{1}{2}\,\mathsf{Scal}^c\cdot\varphi^{\pm} = -\frac{3\gamma^2(n-3)}{2n}\varphi^{\pm}$, i.e., $dT\cdot\varphi^{\pm} = -3\|T\|^2\varphi^{\pm}$, which agrees with [**FI02**, Lem. 10.7].

Example 1.6. Consider a 7-dimensional nearly parallel G_2-manifold. There is a unique ∇^c-parallel spinor φ_0 with $\gamma = -\sqrt{7}\|T\|$ ([**FI02**]) and the scalar curvature $\mathsf{Scal}^g = \frac{27}{2}\|T\|^2$ coincides with the quantity $\frac{9(n-1)}{4n}\gamma^2 = \frac{9(n-1)}{2(9-n)}\|T\|^2$. Thus, φ_0 must be a Killing spinor with Killing number

$$\kappa = 3\gamma/4n = -\frac{3}{4\sqrt{7}}\|T\|\,,$$

which is well-known by [**F+97**, **FI02**]. Moreover, we compute $dT\cdot\varphi_0 = -\frac{3\gamma^2(n-3)}{2n}\varphi_0$, i.e., $dT\cdot\varphi_0 = -6\|T\|^2\varphi_0$, which agrees with the result in [**FI02**, Ex. 5.2].

Let us now agree on the following notation:

$$\begin{array}{rcll}
\mathsf{Ker}\left(\nabla^c\right) & := & \{\varphi\in\Gamma(\Sigma_\gamma)\subset\Gamma(\Sigma) : \nabla^c\varphi\equiv 0\}\,, & \text{(harmonic spinors)}\\
\mathsf{Ker}\left(D^c\right) & := & \{\varphi\in\Gamma(\Sigma_\gamma)\subset\Gamma(\Sigma) : D^c(\varphi)=0\}\,, & \text{(characteristic spinors)}\\
\mathcal{K}^s(M,g)_\zeta & := & \{\varphi\in\Gamma(\Sigma) : \nabla^s_X\varphi=\zeta X\cdot\varphi \ \ \forall\ X\in\Gamma(TM)\}\,, & \text{(Killing spinors with torsion)}\\
\mathcal{K}(M,g)_\kappa & := & \{\varphi\in\Gamma(\Sigma) : \nabla^g_X\varphi=\kappa X\cdot\varphi \ \ \forall\ X\in\Gamma(TM)\}\,, & \text{(real Killing spinors)}
\end{array}$$

where in the third case $s\neq 0$ and $\zeta\neq 0$, and in the fourth case $\kappa\neq 0$.

Notice that any twistor spinor with torsion with respect to $\nabla^c=\nabla^{1/4}$ which is characteristic, that is, $D^c(\varphi)=0$, is in fact ∇^c-parallel, see [**C16b**]. Thus, from now on we are mainly interested in twistors with torsion for some $s\neq 1/4$. In [**C16b**] we proved the following correspondence.

Theorem 1.7. *Let (M^n,g,T) be a compact connected Riemannian spin manifold with $\nabla^c T=0$ and assume that $\varphi\in\Gamma(\Sigma)$ is a non-trivial spinor field such that $\nabla^c\varphi=0$, where $\nabla^c=\nabla^g+\frac{1}{2}T$ is the characteristic connection. Let $\gamma\in\mathbb{R}\backslash\{0\}$ be a non-zero real number. Then, the following conditions are equivalent:*

(a) $\varphi\in\Gamma(\Sigma_\gamma)\cap\mathsf{Ker}(P^s):=\mathsf{Ker}(P^s|_{\Sigma_\gamma})$ with respect to the family $\{\nabla^s : s\in\mathbb{R}\backslash\{1/4\}\}$,

(b) $\varphi\in\mathcal{K}^s(M,g)_\zeta$ with respect to the family $\{\nabla^s : s\in\mathbb{R}\backslash\{0,1/4\}\}$ with $\zeta=3(1-4s)\gamma/4n$,

(c) $\varphi\in\mathcal{K}(M,g)_\kappa$ with $\kappa=3\gamma/4n$.

This shows that for a ∇^c-parallel spinor φ the Riemannian Killing spinor equation $\nabla^g_X\varphi=\kappa X\cdot\varphi$ with Killing number $\kappa:=3\gamma/4n$ for some $\gamma\neq 0$, is equivalent to the KsT equation $(*)$ for some (and thus any) $s\neq 0,\frac{1}{4}$ with Killing number $\zeta:=3(1-4s)\gamma/4n$, and moreover with the twistor equation $(**)$ for some (and thus any) $s\neq 1/4$, under the additional condition $\varphi\in\Sigma_\gamma$.

Now, a combination of our conclusions with [**FI02**, Thm. 3.4] allows us to present the following conclusion (see [**C16b**] for important details that we avoid to present here).

Corollary 1.8. *Let (M^n,g,T) be compact connected Riemannian spin manifold (M^n,g,T), with $\nabla^c T=0$ and positive scalar curvature given by $\mathsf{Scal}^g=\frac{9(n-1)\gamma^2}{4n}$ for some constant $0\neq\gamma\in\mathrm{Spec}(T)$. If the symmetric endomorphism $dT+\frac{1}{2}\Big[\frac{9(n-1)}{4n}\gamma^2-\frac{3}{2}\|T\|^2\Big]$ acts on Σ with non-negative eigenvalues, then the following classes of spinors, if existent, coincide:*

$$\begin{aligned}
\mathsf{Ker}(\nabla^c) &\cong \bigoplus_{\gamma\in\mathrm{Spec}\,(T)}\Big[\Gamma(\Sigma_\gamma)\cap\mathcal{K}(M,g)_{\frac{3\gamma}{4n}}\Big]\\
&\cong \bigoplus_{\gamma\in\mathrm{Spec}\,(T)}\Big[\Gamma(\Sigma_\gamma)\cap\mathcal{K}^s(M,g)_{\frac{3(1-4s)\gamma}{4n}}\Big]\\
&\cong \bigoplus_{\gamma\in\mathrm{Spec}\,(T)}\Big[\mathsf{Ker}(P^s|_{\Sigma_\gamma})\cap\mathsf{Ker}\left(D^c\right)\Big]\,.
\end{aligned}$$

Here, the parameter s takes values in $\mathbb{R}\backslash\{0,1/4\}$ for the third set, and for the last set we have $s\in\mathbb{R}\backslash\{1/4\}$.

As we said, the most representative classes of special structures for which our results presented above make sense, are the 6-dimensional nearly Kähler manifolds and the 7-dimensional nearly parallel G_2-manifolds. We present the corresponding applications below, and more details for this kind of manifolds are given in [**C16b**], and the references therein.

Theorem 1.9. *On a 6-dimensional nearly Kähler manifold (M^6, g, J) endowed with its characteristic connection ∇^c, the following classes of spinor fields coincide:*

(1) TsT with respect to the family $\{\nabla^s : s \in \mathbb{R}\backslash\{1/4\}\}$, lying in $\Sigma_{\pm 2\|T\|}$,

(2) KsT with respect to the family $\{\nabla^s : s \in \mathbb{R}\backslash\{0, 1/4\}\}$, with $\zeta := \mp\frac{(4s-1)}{4}\|T\|$,

(3) Riemannian Killing spinors,

(4) ∇^c-parallel spinors.

Notice that in [**A+13**, Thm. 6.1] the authors proved that for a 6-dimensional nearly Kähler manifold the Killing number with torsion is given by $\mp\frac{\|T\|}{6}$, and this coincides with the statement of our Theorem 1.9 for $s = 5/12$ (as it should be, according to [**A+13**]). In this way, we generalise the result from [**A+13**], by extending the correspondence to any real number $s \neq 0, 1/4$. This also shows the advantage of our approach to this certain kind of TsT or KsT.

The analogous result for the case of weak G_2-manifolds has the form (see [**C16b**] for details).

Theorem 1.10. *On a nearly-parallel G_2-manifold (M^7, g, ω) endowed with its characteristic connection ∇^c, the following classes of spinor fields coincide:*

(1) TsT with respect to the family $\{\nabla^s : s \in \mathbb{R}\backslash\{1/4\}\}$, lying in $\Sigma_{-\frac{7\tau_0}{6}} \equiv \Sigma_{-\sqrt{7}\|T\|}$,

(2) KsT with respect to the family $\{\nabla^s : s \in \mathbb{R}\backslash\{0, 1/4\}\}$, with $\zeta := \frac{(4s-1)\tau_0}{8} = \frac{3(4s-1)\|T\|}{4\sqrt{7}}$,

(3) Riemannian Killing spinors,

(4) ∇^c-parallel spinors.

The rest of the article is mainly devoted to integrability conditions of ∇^c-parallel KsT with respect to the family $\nabla^s = \nabla^g + 2sT$, for some $s \neq 0, 1/4$, or equivalently of ∇^c-parallel TsT for some $s \neq 1/4$, lying in some $\Gamma(\Sigma_\gamma)$, always under the condition $\nabla^c T = 0$. One of the results is the following:

Proposition 1.11. *Assume that $\nabla^c T = 0$ and that (M^n, g, T) is complete and admits a ∇^c-parallel spinor $0 \neq \varphi \in \Sigma_\gamma$ $(\mathbb{R} \ni \gamma \neq 0)$ lying in the kernel $\mathsf{Ker}(P^s)$ for some $s \neq 1/4$. Then, for any $s \in \mathbb{R}$ the following hold*

$$\begin{aligned}
\mathsf{Ric}^s(X) \cdot \varphi &= \frac{6\gamma^2}{n^2}\Big[\frac{6(n-1)(1-4s)^2 + 96s(1-4s) + 16s(3-4s)(n-3)}{16}\Big] X \cdot \varphi, \\
\mathsf{Scal}^s \varphi &= \frac{6\gamma^2}{n}\Big[\frac{6(n-1)(1-4s)^2 + 96s(1-4s) + 16s(3-4s)(n-3)}{16}\Big] \varphi.
\end{aligned}$$

In particular,

(a) (M^n, g) is a compact Einstein manifold with constant positive scalar curvature $\mathsf{Scal}^g = \frac{9(n-1)\gamma^2}{4n}$.
(b) For any $n > 3$, (M^n, g, T) is a strict ∇^c-Einstein manifold with parallel torsion and constant scalar curvature $\mathsf{Scal}^c = \frac{3(n-3)\gamma^2}{n}$. *For $n = 3$, (M^3, g, T) is* Ric^c*-flat.*
(c) (M^n, g, T) is ∇^s-Einstein for any other $s \in \mathbb{R}\backslash\{0, 1/4\}$, i.e., $\mathsf{Ric}^s = \frac{\mathsf{Scal}^s}{n} g$.

Again this result applies to 6-dimensional nearly Kähler manifolds and 7-dimensional weak G_2-manifolds, see [**C16b**]. It was first proved in [**FI02**] that these manifolds are ∇^c-Einstein, and part (b) of our theorem above provides an alternative way to obtain this claim. Moreover, part (c) extends the situation to the whole family ∇^s. Notice also the 3-dimensional Ric^c-flat case corresponds to the 3-sphere, which we treat in many details. For this case we prove that (see [**C16b**] for the notation used in Theorem 1.12).

Theorem 1.12. *There is a one-to-one correspondence between ϵ-Killing spinors on the 3-sphere $(\mathsf{S}^3, g_{\text{can}}, T^\epsilon)$ and Killing spinors with torsion with respect to the family $\nabla^{\epsilon,s}$ for any $s \neq 0, 1/4$, with Killing number $\zeta = \frac{1-4s}{2}$, i.e. $\nabla^{\epsilon,s}_X \varphi_j = \frac{1-4s}{2} X \cdot \varphi_j$, $\forall\, X \in \Gamma(T\,\mathsf{S}^3)$. In particular, a 3-dimensional compact spin manifold (M^3, g, T) satisfying the assumptions of Proposition 1.11, is isometric to $(\mathsf{S}^3, g_{\text{can}}, T^\epsilon)$.*

At this point we need to stress that in general there exist Killing spinors with torsion (KsT) which are not real Killing spinors, and thus manifolds which are not necessarily Einstein can be endowed with them, e.g,. the Heisenberg group, see [**B-B12**, pp. 54–57].

A final contribution of this work is a result about the relation between the two eigenvalue estimates mentioned in the introduction of this subsection. In particular, we prove that

Proposition 1.13. *Let (M^n, g, ∇^c) $(3 < n \leq 8)$ be a compact connected Riemannian spin manifold with $\nabla^c T = 0$ and positive scalar curvature, carrying a spinor field $\varphi \in \Gamma(\Sigma)$ satisfying the equations given in (1.1) for some $0 \neq \gamma \in \mathrm{Spec}(T)$. If $\beta_{\text{tw}}(\gamma) = \beta_{\text{univ}}(\gamma)$ then φ is a real Killing spinor with respect to g, with Killing number $\kappa = 3\gamma/4n$. Conversely, if φ is a real Killing spinor with $\kappa = 3\gamma/4n$ satisfying (1.1), then $\beta_{\text{tw}}(\gamma) = \beta_{\text{univ}}(\gamma)$ identically.*

Example 1.14. For $n = 7$ and for a nearly parallel G_2-manifold (M^7, g, ω) we compute

$$\beta_{\text{tw}}(\gamma) = \frac{7}{54} \mathsf{Scal}^g = \beta_{\text{univ}}(\gamma)\,.$$

For $n = 6$ and a nearly Kähler manifold (M^6, g, J) we compute

$$\beta_{\text{tw}}(\gamma) = \frac{2}{15} \mathsf{Scal}^g = \beta_{\text{univ}}(\gamma)\,,$$

see also [**A+13**, Ex. 6.1] for the second case.

(B) A new $\frac{1}{2}$-Ricci type formula on the spinor bundle and applications. The main motivation of this article is the following observation: For a Riemannian spin manifold (M^n, g) the authors of [**FK00**] introduced a formula that relates the action of the Ricci endomorphism Ric^g on the spinor bundle with the Riemannian Dirac operator D^g. This is given by

$$\frac{1}{2}\,\mathsf{Ric}^g(X)\cdot\varphi = D^g(\nabla^g_X\varphi) - \nabla^g_X(D^g\varphi) - \sum_{j=1}^n e_j\cdot\nabla^g_{\nabla^g_{e_j}X}\varphi\,,$$

referred to as the *Riemannian $\frac{1}{2}$-Ricci type formula.* In [**FK00**] it was shown that this identity is stronger than the Schrödinger-Lichnerowicz formula associated to the Riemann Dirac operator $D^g \equiv D^0$, in the sense that the first formula induces the second one, after a contraction. Moreover, observe that for a (non-trivial) ∇^g-parallel spinor it immediately induces the know Ricci flatness, i.e., $\mathsf{Ric}^g = 0$ identically.

In [**C17**] we extend these results on Riemannian spin manifolds (M^n, g) endowed with the family $\nabla^s = \nabla^g + 2sT$ described above, under the condition $\nabla^c T = 0$, where as usual $\nabla^c = \nabla^{\frac{1}{4}} = \nabla^g + \frac{1}{2}T$. We first prove the following *generalized $\frac{1}{2}$-Ricci type formula.*

Lemma 1.15. (The generalized $\frac{1}{2}$-Ricci type formula, or $\frac{1}{2}$-Ric^s-formula) *Assume that $\nabla^c T = 0$. Then, the Ricci endomorphism $\mathsf{Ric}^s(X)$ satisfies*

$$\begin{aligned}\frac{1}{2}\,\mathsf{Ric}^s(X)\cdot\varphi &= D^s(\nabla^s_X\varphi) - \nabla^s_X(D^s\varphi) - \sum_{j=1}^n e_j\cdot\Big[\nabla^s_{\nabla^s_{e_j}X}\varphi + 4s\nabla^s_{T(X,e_j)}\varphi\Big]\\ &\quad + s(3-4s)(X\lrcorner\sigma_T)\cdot\varphi,\end{aligned}$$

for any arbitrary vector field $X\in\Gamma(TM)$, spinor field $\varphi\in\Gamma(\Sigma)$, and $s\in\mathbb{R}$.

Notice for $s = 0$ the generalized $\frac{1}{2}$-Ricci type formula, reduces to the Riemannian $\frac{1}{2}$-Ricci type formula. A key point of the proof of this identity is that the Ricci endomorphism associated to ∇^s satisfies the following identity

$$\frac{1}{2}\,\mathsf{Ric}^s(X)\cdot\varphi = -\sum_i e_i\cdot\mathcal{R}^s_{X,e_i}\varphi + s(3-4s)(X\lrcorner\sigma_T)\cdot\varphi,$$

for any $X\in\Gamma(TM)$, $\varphi\in\Gamma(\Sigma)$ and $s\in\mathbb{R}$, where

$$\mathcal{R}^s_{X,Y} := [\nabla^s_X, \nabla^s_Y] - \nabla^s_{[X,Y]} : \Gamma(\Sigma)\to\Gamma(\Sigma)$$

is the (spinorial) curvature operator associated to ∇^s. This expression was proved in [**B-B12**, Lem. 1.13] (see also [**A+13**]), and holds only under the assumption $\nabla^c T = 0$.

The new $\frac{1}{2}$-Ric^s-formula has a series of applications, and for convenience we list the most important below.

(i) It induces the corresponding *generalized formula of Schrödinger-Lichnerowicz type*, associated to the Dirac operator D^s (see for example [**FI02**, Thm. 3.1], [**AF04a**, Thm. 6.1] or [**A03**, Thm. 3.2]), under the condition $\nabla^c T = 0$:

$$(D^s)^2(\varphi) = \Delta^s(\varphi) + s(3-4s)dT\cdot\varphi - 4s\mathcal{D}^s(\varphi) + \frac{1}{4}\,\mathsf{Scal}^s\cdot\varphi\,.$$

Here, $\mathcal{D}^s$ is the first-order differetntial operator defined by $\mathcal{D}^s\varphi := \sum_i (e_i\lrcorner T)\cdot\nabla^s_{e_i}\varphi$ and $\Delta^s := (\nabla^s)^*\nabla^s$ denotes the spin Laplace operator associated to ∇^s. Hence, when the torsion form T is ∇^c-parallel we provide a new proof for this fundamental formula, which is different comparing the known proofs (cf. [**FI02**, **A03**, **AF04a**]). Notice the condition $\nabla^c T = 0$ plays an extra role in the proof given in [**C17**] (see pages 3015-3016). This is because under this condition the Ricci tensor Ric^s associated to ∇^s is symmetric, for all $s \in \mathbb{R}$ (see also [**A+13**]), and hence it behaves as the Riemmanian Ricci tensor.

(ii) It provides an alternative, easier, proof of the generalized twistorial $\frac{1}{2}$-Ricci formula that we proved in [**C16b**], see Theorem 1.2 in this thesis.

(iii) It has important applications related to ∇^s-parallel spinors and in particular ∇^c-spinors. We present some of them below.

So, let us explain a few details for the third case, and for the first two cases (i), (ii) we refer to [**C17**].

Looking for ∇^s-parallel spinors, the new $\frac{1}{2}$-Ric^s-identity immediately yields integrability conditions for any member of the family $\{\nabla^s : s \in \mathbb{R}\}$. In particular

Corollary 1.16. *Assume that $\nabla^c T = 0$ and let $\varphi_0 \in \Gamma(\Sigma)$ be a non-trivial spinor field which is parallel with respect to ∇^s, for some $s \in \mathbb{R}$. Then, for the same s and for any $X \in \Gamma(TM)$ the spinor φ_0 must satisfy the following:*

$$\begin{aligned} \mathsf{Ric}^s(X) \cdot \varphi_0 &= 2s(3-4s)(X \lrcorner \sigma_T) \cdot \varphi_0 \,, \\ \mathsf{Scal}^s \cdot \varphi_0 &= -8s(3-4s)\sigma_T \cdot \varphi_0 \,. \end{aligned}$$

Of course, for $s = 0$, that is, ∇^g-parallel spinors, this gives the Ricci flatness of (M, g), while for $s = 1/4$ and the characteristic connection the above formulas reduce to the known integrability conditions for ∇^c-parallel spinors presented in [**FI02**].

Now, when a ∇^c-parallel spinor exists, the $\frac{1}{2}$-Ric^s-identity allows us to describe the action of the endomorphism $\mathsf{Ric}^s(X) : \Gamma(\Sigma) \to \Gamma(\Sigma)$ for any other s. This is one of the most important applications of this identity. In particular, we prove the following

Theorem 1.17. *Consider a Riemannian spin manifold (M^n, g, T) $(n \geq 3)$ endowed with a non-trivial 3-form T, such that $\nabla^c T = 0$, where $\nabla^c := \nabla^g + \frac{1}{2}T$. Assume that φ_0 is a non-trivial ∇^c-parallel spinor field lying in $\Gamma(\Sigma_\gamma)$, for some (constant) $\gamma \in \mathbb{R}$. Then, for any $s \in \mathbb{R}$ and $X \in \Gamma(TM)$ the following holds*

$$\begin{aligned} \mathsf{Ric}^s(X) \cdot \varphi_0 &= \frac{(16s^2-1)}{4} \sum_{j=1}^n T(X, e_j) \cdot (e_j \lrcorner T) \cdot \varphi_0 + \frac{(16s^2+3)}{4}(X \lrcorner \sigma_T) \cdot \varphi_0 \\ &= \mathsf{Ric}^c(X) \cdot \varphi_0 - \frac{(16s^2-1)}{4} S(X) \cdot \varphi_0, \end{aligned}$$

where the action of the endomorphism $S(X)$ is given by

$$\begin{aligned} S(X) \cdot \varphi_0 &= -(X \lrcorner \sigma_T) \cdot \varphi_0 + \sum_{j=1}^n e_j \cdot \big(T(X, e_j) \lrcorner T\big) \cdot \varphi_0 \\ &= \frac{1}{2} \sum_{j=1}^n e_j \cdot (X \lrcorner T) \cdot (e_j \lrcorner T) \cdot \varphi_0 - \frac{3\gamma}{2}(X \lrcorner T) \cdot \varphi_0 - (X \lrcorner \sigma_T) \cdot \varphi_0 \,. \end{aligned}$$

The proof of this remarkable result is long and totally relies on the new $\frac{1}{2}$-Ric^s-formula and the ∇^c-parallelism of T. Its importance is mainly revealed by the given expression of the spinorial action of S. In [**C17**] we present applications of Theorem 1.17 adapted to nearly Kähler manifolds and weak G_2-manifolds, verifying via an alternative method some results about these manifolds obtained in our earlier work [**C16b**]. We also treat Sasakian manifolds for which is well known that their characteristic connection has parallel skew-torsion (for more details on Sasakian manifolds, from our point of view, we refer to [**FI02**, **A06**]). We focus on the 5-dimensional case and based on Theorem 1.17 we prove the following result (see [**C17**] for details on the notation).

Theorem 1.18. *Consider a 5-dimensional simply-connected Sasakian manifold $(M^5, g, \xi, \eta, \phi)$ with its characteristic connection $\nabla^c = \nabla^g + \frac{1}{2}\eta \wedge d\eta = \nabla^g + \eta \wedge F$. Then,*
(1) There exists a ∇^c-parallel spinor $\varphi_1 \in \Sigma^g_{-4}M$, or $\varphi_1 \in \Sigma^g_4 M$, if and only if for any $s \in \mathbb{R}$ the eigenvalues of the Ricci tensor Ric^s are given by

$$\Big\{(6-32s^2), (6-32s^2), (6-32s^2), (6-32s^2), -4(16s^2-1)\Big\}.$$

(2) There exists a ∇^c-parallel spinor $\varphi_0 \in \Sigma^g_0 M$, if and only if for any $s \in \mathbb{R}$ the eigenvalues of the Ricci tensor Ric^s are given by

$$\Big\{-(2+32s^2), -(2+32s^2), -(2+32s^2), -(2+32s^2), -4(16s^2-1)\Big\}.$$

Notice that this result for the special case $s = 1/4$ reduces to a known result by [**FI02**] (obtained however in a different way).

Other contributions presented in [**C17**] are about the differential operator

$$\mathcal{D}^s\varphi = \sum_i (e_i \lrcorner T) \cdot \nabla^s_{e_i}\varphi\,,$$

which is a part of the generalized Schrödinger-Lichnerowicz formula. In particular, we proceed with a first systematic study of this operator, again under the condition $\nabla^c T = 0$ (this 1st-order differential operator was first mentioned in the naturally reductive homogeneous setting by Agricola in [**A03**], and this was the motivation for our study in [**C17**]). One of our results verifies that, under our assumptions, ∇^c-parallel spinors are also eigenspinors of $\mathcal{D}^s$ for any $s \in \mathbb{R}$.

Proposition 1.19. *Consider a Riemannian spin manifold (M^n, g, T) $(n \geq 3)$ with $\nabla^c T = 0$, where $\nabla^c := \nabla^g + \frac{1}{2}T$ is the metric connection with skew-torsion $T \neq 0$. Assume that $\varphi_0 \in \Gamma(\Sigma_\gamma)$ is a non-trivial ∇^c-parallel spinor and $\gamma \in \mathrm{Spec}(T)$ is an eigenvalue of T, where T is viewed as a symmetric endomorphism on the spinor bundle. Then, φ_0 is an eigenspinor of the operator $\mathcal{D}^s$ for any $s \in \mathbb{R}$,*

$$\mathcal{D}^s(\varphi_0) = -\frac{(4s-1)}{4}\Big[T^2 + 2\|T\|^2\Big] \cdot \varphi_0 = -\frac{(4s-1)}{4}\Big[\gamma^2 + 2\|T\|^2\Big]\varphi_0.$$

Additional conclusions are presented in the final section of [**C17**].

(C) Invariant connections and ∇-Einstein structures on isotropy irreducible spaces.

In [C+19] we present a series of new results related to invariant connections and their torsion type, on compact, effective, naturally reductive Riemannian manifolds. In particular, we examine both the symmetric and non-symmetric case and we develop some theory available for handling the classification of all G-invariant metric connections, with respect to a naturally reductive metric. Then we apply this theory to classify invariant affine connections on (compact) non-symmetric strongly isotropy irreducible homogeneous Riemannian manifolds. Hence, this work can be seen as a continuation of the works of Laquer on the classification of invariant affine connections on compact Lie groups and symmetric spaces ([L92a, L92b]), and this was part of our motivation.

Notice that any (effective) non-symmetric SII space $M = G/K$ admits a family of invariant metric connections induced by the $\mathsf{Ad}(K)$-invariant bilinear map $\eta^\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ with $\eta^\alpha(X, Y) := \frac{1-\alpha}{2}[X, Y]_{\mathfrak{m}}$, where $\mathfrak{m}$ is the B-orthogonal reductive complement of $\mathfrak{k} = \mathsf{Lie}(K)$ in $\mathfrak{g} = \mathsf{Lie}(G)$. In full details, this family, which we call the *Lie bracket family*, has the form

$$\nabla^\alpha_X Y = \nabla^{\mathfrak{m}}_X Y + \eta^\alpha(X, Y) = \nabla^g_X Y + \frac{\alpha}{2} T^{\mathfrak{m}}(X, Y),$$

where $\nabla^{\mathfrak{m}}$ denotes the canonical connection associated to $\mathfrak{m}$ and ∇^g is the Levi-Civita connection of the Killing metric.[2] Hence, its torsion is an invariant 3-form on M, given by $T^\alpha = \alpha \cdot T^{\mathfrak{m}}$, where $T^{\mathfrak{m}}$ is the torsion of $\nabla^{\mathfrak{m}}$.

An important point of the study in [C+19] is the conclusion that the family ∇^α does *not* exhaust in general all G-invariant metric connections on such cosets G/K, even with skew-torsion. In particular, for the classification of invariant connections on $M = G/K$ one needs to decompose the modules $\Lambda^2(\mathfrak{m})$ and $\mathsf{Sym}^2(\mathfrak{m})$ into irreducible submodules. For such a procedure we mainly use the LiE program. As a result, for any effective non-symmetric (compact) SII homogeneous Riemannian manifold $(M = G/K, g = -B|_{\mathfrak{m}})$ we state the dimension of the space $\mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, see Tables 4 and 5 in [C+19]. In addition to this, for any such homogeneous space we present the space of G-invariant torsion-free connections and classify the dimension of the space of G-invariant metric connections. Moreover, we state the multiplicity of the (real) trivial representation inside the space $\Lambda^3(\mathfrak{m})$ of 3-forms. This last step yields finally the presentation of the subclass of G-invariant metric connections with skew-torsion. Note that all these desired multiplicities were also obtained in [Cℓ01], up to some errors/omissions, that we correct. We summarize our results as follows:

Theorem 1.20. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective non-symmetric SII space. Then:*
(i) The family $\{\nabla^\alpha : \alpha \in \mathbb{R}\}$ exhausts all G-invariant affine or metric connections on $M = G/K$, if and only if $G = \mathsf{Sp}(n)$, or $M = G/K$ is $\mathsf{G}_2/\mathsf{SO}(3)$, or one of the manifolds

$\mathsf{SO}(14)/\mathsf{Sp}(3)$,	$\mathsf{SO}(4n)/\mathsf{Sp}(n)\times\mathsf{Sp}(1)$ $(n \geq 2)$,	$\mathsf{SO}(7)/\mathsf{G}_2$,	$\mathsf{SO}(16)/\mathsf{Spin}(9)$,
$\mathsf{F}_4/(\mathsf{G}_2\times\mathsf{SU}(2))$,	$\mathsf{E}_7/(\mathsf{G}_2\times\mathsf{Sp}(3))$,	$\mathsf{E}_7/(\mathsf{F}_4\times\mathsf{SU}(2))$,	$\mathsf{E}_8/(\mathsf{F}_4\times\mathsf{G}_2)$.

[2] The parameter α can be a real or complex number, depending on the type of the isotropy representation $\mathfrak{m}$.

The same family exhausts also all $\mathsf{SU}(2q)$*-invariant metric connections on the homogeneous space* $\mathsf{SU}(2q)/\mathsf{SU}(2)\times\mathsf{SU}(q)$ $(q\geq 3)$*, but not all the* $\mathsf{SU}(2q)$*-invariant affine connections.*

(ii) The family $\{\nabla^\alpha : \alpha\in\mathbb{C}\}$ *exhausts all* G*-invariant affine or metric connections on* $M=G/K$*, if and only if* $M=G/K$ *is one of the manifolds*

$$\begin{array}{llll}\mathsf{SO}(8)/\mathsf{SU}(3), & \mathsf{G}_2/\mathsf{SU}(3), & \mathrm{F}_4/(\mathsf{SU}(3)\times\mathsf{SU}(3)), & \mathsf{E}_6/(\mathsf{SU}(3)\times\mathsf{SU}(3)\times\mathsf{SU}(3)),\\ \mathsf{E}_7/(\mathsf{SU}(3)\times\mathsf{SU}(6)), & \mathsf{E}_8/\mathsf{SU}(9), & \mathsf{E}_8/(\mathsf{E}_6\times\mathsf{SU}(3)). & \end{array}$$

Notice for the 6-sphere $\mathsf{S}^6=\mathsf{G}_2/\mathsf{SU}(3)$ and the 7-sphere $\mathsf{S}^7=\mathsf{SO}(7)/\mathsf{G}_2$ these results were known from a previous work of the author, see [**C16a**]. For invariant metric connections different from ∇^α, we prove that

Theorem 1.21. *Let* $(M=G/K, g=-B|_{\mathfrak{m}})$ *be an effective non-symmetric SII space which admits at least one invariant metric connection, different from the Lie bracket family. Then,* $M=G/K$ *is isometric to a space given in Table 2 in* [**C+19**]*. In this table we present the dimensions of the spaces* $\mathsf{Hom}_K(\mathfrak{m},\Lambda^2\mathfrak{m})$ *and* $(\Lambda^3\mathfrak{m})^K$*, which respectively parametrize the space of invariant metric connections and the space of invariant metric connections with totally skew-symmetric torsion. In particular:*
(i) Any homogeneous space in Table 2 of [**C+19**] *whose isotropy representation is of real type and which is not isometric to* $\mathsf{SO}(10)/\mathsf{Sp}(2)$*, admits a 2-dimensional space of* G*-invariant metric connections with skew-torsion. For* $\mathsf{SO}(10)/\mathsf{Sp}(2)$*, the unique family of* $\mathsf{SO}(10)$*-invariant metric connections with skew-torsion is given by* ∇^α $(\alpha\in\mathbb{R})$*. However, the space of all* $\mathsf{SO}(10)$*-invariant metric connections is 2-dimensional.*
(ii) Any homogeneous space in Table 2 of [**C+19**] *whose isotropy representation is of complex type, admits a 6-dimensional space of* G*-invariant metric connections and a 4-dimensional subspace of* G*-invariant metric connections with skew-torsion.*

For invariant connections induced by some non-trivial element $\mu\in\mathrm{Hom}_K(\mathsf{Sym}^2\,\mathfrak{m},\mathfrak{m})$, we prove the following:

Theorem 1.22. *Let* $(M=G/K, g=-B|_{\mathfrak{m}})$ *be an effective non-symmetric SII space, which admits at least one invariant affine connection* ∇^μ*, induced by some non-trivial element* $\mu\in\mathsf{Hom}_K(\mathsf{Sym}^2\,\mathfrak{m},\mathfrak{m})$*. Then:*
(i) If the associated isotropy representation is of real type, then $M=G/K$ *is isometric to a manifold in Table 3 of* [**C+19**]*. In this table,* s *encodes the dimension of the module* $\mathsf{Hom}_K(\mathsf{Sym}^2\,\mathfrak{m},\mathfrak{m})$*, which parametrizes the space of such invariant connections.*

(ii) If the associated isotropy representation is of complex type, then $M=G/K$ *is isometric to one of the manifolds* $\mathsf{SO}(n^2-1)/\mathsf{SU}(n)$ $(n\geq 4)$ *or* $\mathsf{E}_6/\mathsf{SU}(3)$*, where the dimension of the space of such invariant connections is 2 and 4, respectively.*

(iii) The G*-invariant connection* ∇^μ *does not preserve the Killing metric* $g=-B|_{\mathfrak{m}}$*. Thus,* ∇^μ *is not metric with respect to any* G*-invariant metric.*

Now, a combination of these three theorems yields the desired dimension of the space of all G-invariant affine connections for any non-symmetric (compact) SII space $M=G/K$, which we denote by

$$\mathcal{N} := \dim_{\mathbb{R}} \mathcal{A}ff_G(F(G/K)) = \dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}).$$

We refer to the Tables 4 and 5 in [**C+19**], where the number $\mathcal{N}$ is explicitly indicated.

We also compute the dimension of the space of invariant torsion-free connections on a non-symmetric SII space $M = G/K$, denoted by $\mathcal{A}ff^0_G(F(G/K))$. We prove that

Corollary 1.23. *The classification of non-symmetric compact SII spaces which admit new invariant torsion-free connections, in addition to the Levi-Civita connection, is given by the manifolds presented in Theorem 1.22. In particular, for a homogeneous space in Table 3 of* [**C+19**], *we have* $\dim_{\mathbb{R}} \mathcal{A}ff^0_G(F(G/K)) = \mathbf{s}$, *and for the almost complex homogeneous spaces in Theorem 1.22 we have* $\dim_{\mathbb{R}} \mathcal{A}ff^0_G(F(G/K)) = 2$ *or* 4, *respectively.*

For long period it was believed that on a compact Lie group endowed with a bi-invariant metric, any bi-invariant metric connection has skew-torsion (see [**C+19**] for details). We prove that this conclusion is wrong. In particular in [**C+19**, Thm. 3.15] we show that Lie group $\mathsf{U}(n)$ $(n \geq 3)$ endowed with a bi-invariant metric, admits a class of bi-invariant metric connections whose torsion is not a 3-form, but of vectorial type, and this provides another contribution of this article.

In the final section of [**C+19**] we focus on the classification of invariant ∇-Einstein structures with skew-torsion on all compact non-symmetric SII homogeneous spaces. This occurs as a direct application of the first part of the paper about the classification of invariant metric connections with skew-torsion on such homogeneous spaces, and allows us to compute the dimension of the space of such structures for any coset in our classification. The particular statements are presented in Theorems C, D and E in the introduction of [**C+19**] and the proofs are given in the final section, where we refer for more details.

(D) Homogeneous 8-manifolds admitting invariant $\mathsf{Spin}(7)$-structures. In [**Aℓ+20**] we study simply connected compact homogeneous 8-manifolds admitting invariant $\mathsf{Spin}(7)$-structures and classify all canonical presentations G/H of such spaces, with G simply connected. The motivation for the research in [**Aℓ+20**] was the known classification of *invariant* G_2-structures ([**LM12**, **R10**]). In particular, a similar classification for $\mathsf{Spin}(7)$, as in G_2-case was unknown.

Recall that every compact, simply connected, homogeneous space M admits a *canonical presentation*, which means that we can identify M with a homogenous presentation G/H, where G is a compact, connected, simply connected, semisimple Lie group and H is a closed connected subgroup of G. All compact, simply connected, homogeneous 8-dimensional manifolds G/H of a compact, connected, simply connected Lie group G were classified in [**K88**], and in [**Aℓ+20**] we rely on this classification. On the other hand, it is know that $\mathsf{Spin}(7)$-structures are topologically obstructed (see [**LM89**, **I04**, **Aℓ+20**]). Thus, in combination with [**K88**], a topological examination allows us to present the following

Theorem 1.24. *(a)The canonical presentations of all compact, simply connected, non-symmetric almost effective homogeneous spaces admitting a* $\mathsf{Spin}(7)$*-structure are exhausted by*

- $M_1 = \dfrac{\mathsf{SU}(3)}{\{e\}}$;
- $M_2 = C_{k,\ell,m} := \dfrac{\mathsf{SU}(2)\times\mathsf{SU}(2)\times\mathsf{SU}(2)}{\mathsf{U}(1)_{k,\ell,m}}, \quad k \geq \ell \geq m \geq 0,\ k > 0,\ \gcd(k,\ell,m) = 1$;
- $M_3 = \dfrac{\mathsf{SU}(2)\times\mathsf{SU}(2)\times\mathsf{SU}(2)}{\Delta\,\mathsf{SU}(2)} \times \dfrac{\mathsf{SU}(2)}{\mathsf{U}(1)}$;
- $M_4 = \dfrac{\mathsf{SU}(3)}{\mathsf{SU}(2)} \times \mathsf{SU}(2)$;

As smooth manifolds, the spaces M_2 and M_3 are diffeomorphic to $\mathsf{S}^3\times\mathsf{S}^3\times\mathsf{S}^2$, while M_4 is diffeomorphic to $\mathsf{S}^5\times\mathsf{S}^3$.
(b) The compact simply connected symmetric spaces admitting a $\mathsf{Spin}(7)$-structure are exhausted by $\mathsf{SU}(3)$, $\mathbb{HP}^2$, $\mathrm{Gr}_2(\mathbb{C}^4)$, the exceptional Wolf space $\dfrac{\mathsf{G}_2}{\mathsf{SO}(4)}$ and the products $\mathsf{S}^3\times\mathsf{S}^3\times\mathsf{S}^2$ and $\mathsf{S}^5\times\mathsf{S}^3$.

The manifold $C_{k,\ell,m}$ appearing in the above theorem is a torus bundle over the homogeneous Kähler-Einstein manifold $(\mathsf{SU}(2)/\mathsf{U}(1))^{\times 3}$, and hence a non-Kähler C-space (see [**CS21**] for details on such spaces). When $m = 0$, $C_{k,\ell,0}$ is the direct product of S^3 with the total space of a circle bundle over $\mathsf{S}^2\times\mathsf{S}^2$. Furthermore, $C_{1,0,0} = \mathsf{Spin}(4)\times\frac{\mathsf{SU}(2)}{\mathsf{U}(1)}$.
On the other hand, the homogeneous space $(\mathsf{SU}(3)/\mathsf{SU}(2))\times\mathsf{SU}(2)$ is an example of a Calabi–Eckmann manifold. This is a torus bundle over $\mathbb{CP}^2\times\mathbb{CP}^1$, and hence also a non-Kähler C-space.
Using general properties of symmetric spaces, we see that there are no *invariant* $\mathsf{Spin}(7)$-structures on such manifolds. Combining this fact with a case-by-case analysis of the homogeneous spaces appearing in the above theorem we obtain the following classification result.

Theorem 1.25. *The canonical presentations of all compact, simply connected, almost effective homogeneous spaces admitting an invariant $\mathsf{Spin}(7)$-structure are exhausted by $\frac{\mathsf{SU}(3)}{\{e\}}$, the infinite family $C_{k,\ell,m}$, for $k = \ell + m$, and the Calabi-Eckmann manifold $\frac{\mathsf{SU}(3)}{\mathsf{SU}(2)}\times\mathsf{SU}(2)$.*

It is remarkable that there are just a few examples of 8-dimensional compact simply connected homogeneous spaces admitting invariant $\mathsf{Spin}(7)$-structures. This is different from the case of G_2-structures, where examples of this type are abundant (see [**R10**], and compare with the classification of compact almost effective homogeneous 7-manifolds given in [**Aℓ+19**]).
Notice in [**Aℓ+20**] we also describe the invariant Riemannian metrics and the invariant differential forms for the homogeneous spaces $C_{k,\ell,m}$, with $k > \ell > m > 0$, and $(\mathsf{SU}(3)/\mathsf{SU}(2))\times\mathsf{SU}(2)$. This allows us to obtain a 5-parameter family of invariant $\mathsf{Spin}(7)$-structures of mixed type (in terms of [**F86**]) on both of them (the case of $\mathsf{SU}(3)$ was known

by [**F86**]). In particular, for both spaces we show that there exists an invariant $\mathsf{Spin}(7)$-structure Φ inducing the normal metric and whose characteristic connection ∇^c (introduced in [**I04**]), coincides with the canonical connection corresponding to the naturally reductive structure induced by the induced invariant metric g_Φ. Hence in this case ∇^c has parallel torsion. Notice that the results of this paper are also useful to study compact 8-manifolds admitting other types of special structures, e.g., invariant $\mathsf{PSU}(3)$-structures, which is a work under preparation.

(E) Differential geometry of $\mathsf{SO}^*(2n)$-type structures. The article [**C+22a**] provides the first systematic study of $4n$-dimensional manifolds M $(n > 1)$ admitting a reduction of the frame bundle to the Lie subgroups $\mathsf{SO}^*(2n)$, or $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$ of $\mathsf{GL}(4n, \mathbb{R})$, and is the first one in a series of articles devoted to such G-structures (see [**C+22b**, **C+23b**]). Here, $\mathsf{SO}^*(2n) = \mathsf{GL}(n, \mathbb{H}) \cap \mathsf{Sp}(4n, \mathbb{R})$ denotes the quaternionic real form of $\mathsf{SO}(2n, \mathbb{C})$, and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$ denotes the Lie group $\mathsf{SO}^*(2n) \times_{\mathbb{Z}_2} \mathsf{Sp}(1)$. We show that these G-structures arise from almost symplectic forms ω which are H-Hermitian, respectively Q-Hermitian, where H denotes an almost hypercomplex structure and $Q \subset \mathsf{End}(TM)$ an almost quaternionic structure. Hence, it is natural to refer to such G-structures by the terms *almost hypercomplex skew-Hermitian structures*, denoted by (H, ω), and *almost quaternionic skew-Hermitian structures*, denoted by (Q, ω), respectively. This terminology is also motivated by the discussion in [**H90**] on the eight types of inner product spaces.

The main goal was to establish the symplectic analogue of almost hypercomplex-Hermitian and almost quaternion-Hermitian geometries, and derive the intrinsic torsion submodules of the corresponding G-structures. Notice the topic in its full generality was missing from the literature, and only details for the integrable case (torsion-free case) was known by previous works of Schwachhöfer ([**S01a**, **S01b**]) and Cahen-Schwachhöfer ([**CS09**]). Hence the article [**C+22a**] submitted in this thesis, in combination with the rest in this series [**C+22b**, **C+23b**], aims to fill this gap.

To summarize, in [**C+22a**] we explore the underlying geometry of compatible pairs (H, ω) and (Q, ω), as defined above. One of our results certifies the existence of three pseudo-metric tensors g_I, g_J, g_K, which are of signature $(2n, 2n)$ (but not of Norden type). These tensors are globally defined on almost hypercomplex skew-Hermitian manifolds (M, H, ω), but they exist only locally in the case of almost quaternionic skew-Hermitian manifolds (M, Q, ω). However, using them we can introduce the *fundamental 4-tensor* associated to such geometric structures, given by

$$\Phi := g_I \odot g_I + g_J \odot g_J + g_K \odot g_K = \mathsf{Sym}(g_I \otimes g_I + g_J \otimes g_J + g_K \otimes g_K)\,.$$

This symmetric 4-tensor is globally defined for both cases, and establishes the counterpart of the known fundamental 4-form Ω on an almost hypercomplex/quaternionic-Hermitian manifold (see [**S86**, **S91**, **AℓM96**]). We thoroughly investigate this tensor in our second article [**C+22b**].

One of the main contributions in [**C+22a**] is the presentation of (adapted) $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-connections, denoted by $\nabla^{H,\omega}$ and $\nabla^{Q,\omega}$, respectively, and moreover the description of the intrinsic torsion of $\mathsf{SO}^*(2n)$- or $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures (given in a convenient way, in terms of the EH-formalism of Salamon [**S86**]). We thus compute the (second) Spencer cohomology $\mathcal{H}^{0,2}(\mathfrak{so}^*(2n))$ and $\mathcal{H}^{0,2}(\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1))$ associated to the

Lie algebras $\mathfrak{so}^*(2n)$ and $\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$, respectively, and present the number of algebraic types of such geometric structures. We obtain the following (see [C+22a, Corol. 4.15] for details)

Theorem 1.26. *For $n > 3$ the intrinsic torsion module corresponding to $\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$ decomposes into five $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-irreducible submodules,*

$$\mathcal{H}(\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)) \cong \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4 \oplus \mathcal{X}_5 \,.$$

Hence, there are five pure types of $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures and a totality of 2^5 algebraic types.

Geometric interpretations of some of these modules are given in Proposition 5.5 of [C+22a].

On the other hand, the case for the Lie group $\mathsf{SO}^*(2n)$ is rather complicated, due to the appearance of multiplicities. Here we prove that (see [C+22a, Corol. 4.17] for details)

Theorem 1.27. *For $n > 3$ the number of algebraic types of $\mathsf{SO}^*(2n)$-structures is equal to 2^{10}. Moreover, there are seven special $\mathsf{Sp}(1)$-invariant classes $\mathcal{X}_1, \ldots, \mathcal{X}_7$, determined in terms of $\mathsf{Sp}(1)$-invariant conditions.*

For the low-dimensional cases $n = 2, 3$, we finally show that both structures under investigation include some extra algebraic types.

As for the adapted connections $\nabla^{H,\omega}$ and $\nabla^{Q,\omega}$ are obtained by using the Obata connection ∇^H associated to any almost hypercomplex structure H, and an Oproiu connection ∇^Q associated to the almost quaternionic structure Q, see Theorem 4.8 and Theorem 4.12 in [C+22a] for the explicit description of these adapted connections. Based on our intrinsic torsion decompositions, we also prove that these connections are the unique minimal connections for our structures, with respect to certain normalization conditions given in [C+22a, Theorem 5.3]. We then rely on these minimal connections to answer the question of equivalence of such G-structures (see Section 5.2 of [C+22a]).

Another contribution of [C+22a] is the classification of symmetric spaces K/L with K semisimple admitting an invariant torsion-free $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structure. Based on the classification of the pseudo-Wolf spaces, given by Alekseevsky-Cortés in [AℓC05], and using an older result by [G13] we prove that

Theorem 1.28. *The symmetric space $\mathsf{SO}^*(2n+2)/\,\mathsf{SO}^*(2n)\,\mathsf{U}(1)$ and the pseudo-Wolf spaces*

$$\mathsf{SU}(2+p,q)/(\mathsf{SU}(2)\,\mathsf{SU}(p,q)\,\mathsf{U}(1))\,, \qquad \mathsf{SL}(n+1,\mathbb{H})/(\mathsf{GL}(1,\mathbb{H})\,\mathsf{SL}(n,\mathbb{H}))$$

are the only (up to covering) symmetric spaces K/L with K semisimple, admitting an invariant torsion-free quaternionic skew-Hermitian structure (Q,ω). In particular, the corresponding canonical connections on any of these symmetric spaces coincides with the associated minimal quaternionic skew-Hermitian connection $\nabla^{Q,\omega}$.

We finally present a local construction of torsion-free $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures with special symplectic holonomy, i.e., $T^{Q,\omega} = 0$ and $\mathsf{Hol}(\nabla^Q, \omega) = \mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$, as a direct application of a result by Cahen-Schwachhöfer ([CS09]), see Proposition 6.3 in [C+22a].

At this point we mention that the investigation of the particular contributions of the different intrinsic torsion components in the obstruction of the integrability of H, Q and ω, is described in the second paper devoted to this kind of geometries (see [**C+22b**]). In the same article we provide a plethora of examples of manifolds admitting such structures in the non-integrable case, and in particular we realize some of the intrinsic torsion types.

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Invariant connections and ∇-Einstein structures on isotropy irreducible spaces

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ABSTRACT

This paper is devoted to a systematic study and classification of invariant affine or metric connections on certain classes of naturally reductive spaces. For any non-symmetric, effective, strongly isotropy irreducible[2] homogeneous Riemannian manifold $(M = G/K, g)$, we compute the dimensions of the spaces of G-invariant affine and metric connections. For such manifolds we also describe the space of invariant metric connections with skew-torsion. For the compact Lie group U_n we classify all bi-invariant metric connections, by introducing a new family of bi-invariant connections whose torsion is of vectorial type. Next we present applications related with the notion of ∇-Einstein manifolds with skew-torsion. In particular, we classify all such invariant structures on any non-symmetric strongly isotropy irreducible homogeneous space.

0. Introduction

Motivation. Given ahomogeneous space $M = G/K$ with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, a G-invariant affine connection ∇ is nothing but a connection in the frame bundle $F(M) = G \times_K \mathrm{GL}(\mathfrak{m})$ of M which is also G-invariant. The first studies of invariant connections were performed by Nomizu [36], Wang [44] and Kostant [30] during the fifties. After that, homogeneous connections on principal bundles have attracted the interest of both mathematicians and physicists, with several different perspectives; for example Cartan connections and parabolic geometries [16], Lie triple systems and Yamaguti–Lie algebras [23,12], Yang–Mills and gauge theories [27,31], etc. From another point of view, invariant connections are crucial in the holonomy theory of naturally reductive spaces and Dirac operators, mainly due to the special properties of the *canonical connection* (or the characteristic connection in terms of special structures, see [29,38,37,20,1,4,6]).

According to [44], given a G-homogeneous principal bundle $P \to G/K$ with structure group U, there is a bijective correspondence between G-invariant connections on P and certain linear maps $\Lambda : \mathfrak{g} \to \mathfrak{u}$, where $\mathfrak{g}$, $\mathfrak{u}$ are the Lie algebras of G and U, respectively. Wang's correspondence was successfully used by Laquer [32,33] during the nineties to describe the set of

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2 We will use the abbreviation SII for *strongly isotropy irreducible*.

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Table 1
Invariant connections on compact irreducible symmetric spaces due to [32,33].

Type I	$M = G/K$	Invariant connections $\mathcal{A}\!ff_G(F(M))$
AI	SU_n/SO_n $(n \geq 3)$	1-dimensional family
AII	SU_{2n}/Sp_n $(n \geq 3)$	1-dimensional family
EIV	E_6/F_4	1-dimensional family
	All the other cases	Canonical connection $\equiv$ Levi-Civita connection
Type II	$M = (G \times G)/\Delta G$	Bi-invariant connections $\mathcal{A}\!ff_{G\times G}(F(M))$
	SU_n $(n \geq 3)$	2-dimensional family
	All the other simple Lie groups	1-dimensional family (inducing the flat ± 1-connections)

invariant affine connections, denoted by $\mathcal{A}\!ff_G(F(G/K))$, on compact irreducible Riemannian symmetric spaces $M = G/K$. For most cases, Laquer proved that $\mathcal{A}\!ff_G(F(G/K))$ consists of the canonical connection (simple Lie groups admit a line of canonical connections), except for a few cases where new 1-parameter families arise, see Table 1. By contrast, much less is known about invariant connections on *non-symmetric* homogeneous spaces, even in the isotropy irreducible case. For example, the first author in [17], considered invariant connections on manifolds G/K diffeomorphic to a symmetric space, which however do not induce a symmetric pair (G, K), e.g. $G_2/SU_3 \cong S^6$ and $Spin_7/G_2 \cong S^7$. There, it was shown that the space of G_2-invariant affine or metric connections on the sphere $S^6 = G_2/SU(3)$ is 2-dimensional, while the space of $Spin_7$-invariant affine or metric connections on the 7-sphere $S^7 = Spin_7/G_2$ is 1-dimensional. This is a remarkable result, since the only SO_7- (resp. SO_8)-invariant affine (or metric) connection on the symmetric space $S^6 = SO_7/SO_6$ (resp. $S^7 = SO_8/SO_7$) is the canonical connection.

Motivated by this simple result, in this article we classify invariant affine connections on *(compact) non-symmetric strongly isotropy irreducible homogeneous Riemannian manifolds*. A connected effective homogeneous space G/K is called isotropy irreducible if K acts irreducibly on $T_o(G/K)$ via the isotropy representation. If the identity component K_0 of K also acts irreducibly on $T_o(G/K)$, then G/K is called *strongly isotropy irreducible*. Obviously, any strongly isotropy irreducible space is also isotropy irreducible but the converse is false, see [13]. Non-symmetric strongly isotropy irreducible homogeneous spaces were originally classified by Manturov (see for example [13,45]) and were later studied by Wolf [46] and others. Any SII space admits a unique invariant Einstein metric, the so-called Killing metric and in the non-compact case such a manifold is a symmetric space of non-compact type. In fact, SII spaces share many properties with symmetric spaces and indeed, any irreducible (as Riemannian manifold) symmetric space is strongly isotropy irreducible. A conceptual relationship between symmetric spaces and SII spaces was explained in [45]. More recently, isotropy irreducible homogeneous spaces endowed with their canonical connection ∇^c was shown to have a special relationship with geometric structures with torsion (see [20,19]).

Outline and classification results. After recalling preliminaries in Section 1 about (invariant) metric connections and their torsion types, in Section 2 we fix a reductive homogeneous space $(M = G/K, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m})$ and introduce the notion of generalized derivations of a tensor $F : \otimes^p \mathfrak{m} \to \mathfrak{m}$. When F is $\mathrm{Ad}(K)$-invariant and $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ is a K-intertwining map, we prove that μ induces a generalized derivation of F if and only if F is ∇^μ-parallel, where ∇^μ is the invariant connection on M associated to μ (Theorem 2.5). Moreover, we conclude that for an invariant tensor field F the operation induced by a generalized derivation $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, coincides with the covariant derivative $\nabla^\mu F$ (Corollary 2.6). Then we consider derivations on $\mathfrak{m}$ and provide necessary and sufficient conditions for their existence (Theorem 2.8), generalizing results from [17].

Next, in Section 3 we present a series of new results related to invariant connections and their torsion type, on compact, effective, naturally reductive Riemannian manifolds. In particular, we examine both the symmetric and non-symmetric case and we develop some theory available for the classification of all G-invariant metric connections, with respect to a naturally reductive metric (see Lemma 3.9, Lemma 3.12, Theorem 3.13). In fact, in this way we correct some wrong conclusions given in [6,17]. For example, for the compact Lie group U_n $(n \geq 3)$ endowed with a bi-invariant metric we present a class of bi-invariant metric connections whose torsion is not a 3-form, but of vectorial type (Theorem 3.15, Proposition 3.20).

In Section 4 we focus on (compact) non-symmetric, strongly isotropy irreducible homogeneous Riemannian manifolds $(M = G/K, g = -B|_{\mathfrak{m}})$ with aim the classification of all G-invariant affine or metric connections. We always work with an effective G-action, and based on our previous results on effective naturally reductive spaces we first prove that a G-invariant metric connection on $(M = G/K, g = -B|_{\mathfrak{m}})$ cannot admit a component of vectorial type (Proposition 4.1). Then, in the spin case we describe an application about the formal self-adjointness of Dirac operators associated to invariant metric connections on such types of homogeneous spaces (Corollary 4.4).

Notice now that any (effective) non-symmetric SII space $M = G/K$ admits a family of invariant metric connections induced by the $\mathrm{Ad}(K)$-invariant bilinear map $\eta^\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ with $\eta^\alpha(X, Y) := \frac{1-\alpha}{2}[X, Y]_{\mathfrak{m}}$. In full details, this family, which we call the *Lie bracket family*, has the form

$$\nabla^\alpha_X Y = \nabla^c_X Y + \eta^\alpha(X, Y) = \nabla^g_X Y + \frac{\alpha}{2} T^c(X, Y),$$

where ∇^c denotes the canonical connection associated to $\mathfrak{m}$ and ∇^g the Levi-Civita connection of the Killing metric.[2] Hence, its torsion is an invariant 3-form on M, given by $T^\alpha = \alpha \cdot T^c$, where T^c is the torsion of ∇^c (see [1,17]). However, we will show that in general the family ∇^α does not exhaust all G-invariant metric connections, even with skew-torsion. In particular, for the classification of invariant connections on $M = G/K$ one needs to decompose the modules $\Lambda^2(\mathfrak{m})$ and $\mathrm{Sym}^2(\mathfrak{m})$ into irreducible submodules. For such a procedure we mainly use the LiE program,[3] but also provide examples of how such spaces can be treated only by pure representation theory arguments, without the aid of a computer (see Section 4.5). As a result, for any effective non-symmetric (compact) SII homogeneous Riemannian manifold $(M = G/K, g = -B|_{\mathfrak{m}})$ we state the dimension of the space $\mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ (see Theorem 4.7, Tables 4, 5). In addition to this, for any such homogeneous space we present the space of G-invariant torsion-free connections and classify the dimension of the space of G-invariant metric connections. Moreover, we state the multiplicity of the (real) trivial representation inside the space $\Lambda^3(\mathfrak{m})$ of 3-forms. This last step yields finally the presentation of the subclass of G-invariant metric connections with skew-torsion. Note that all these desired multiplicities were also obtained in [19], up to some errors/omissions, see Remark 4.5 and Table 4 for corrections. We summarize our results as follows:

Theorem A.1. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective non-symmetric SII space. Then:*
(i) The family $\{\nabla^\alpha : \alpha \in \mathbb{R}\}$ exhausts all G-invariant affine or metric connections on $M = G/K$, if and only if $G = \mathrm{Sp}_n$, or $M = G/K$ is one of the manifolds

$$\begin{array}{lllll} \mathrm{SO}_{14}/\mathrm{Sp}_3, & \mathrm{SO}_{4n}/\mathrm{Sp}_n \times \mathrm{Sp}_1\ (n \geq 2), & \mathrm{SO}_7/\mathrm{G}_2, & \mathrm{SO}_{16}/\mathrm{Spin}_9, & \mathrm{G}_2/\mathrm{SO}_3, \\ \mathrm{F}_4/(\mathrm{G}_2 \times \mathrm{SU}_2), & \mathrm{E}_7/(\mathrm{G}_2 \times \mathrm{Sp}_3), & \mathrm{E}_7/(\mathrm{F}_4 \times \mathrm{SU}_2), & \mathrm{E}_8/(\mathrm{F}_4 \times \mathrm{G}_2). & \end{array}$$

The same family exhausts also all SU_{2q}-invariant metric connections on the homogeneous space $\mathrm{SU}_{2q}/\mathrm{SU}_2 \times \mathrm{SU}_q$ ($q \geq 3$), but not all the SU_{2q}-invariant affine connections.
(ii) The family $\{\nabla^\alpha : \alpha \in \mathbb{C}\}$ exhausts all G-invariant affine or metric connections on $M = G/K$, if and only if $M = G/K$ is one of the manifolds

$$\begin{array}{llll} \mathrm{SO}_8/\mathrm{SU}_3, & \mathrm{G}_2/\mathrm{SU}_3, & \mathrm{F}_4/(\mathrm{SU}_3 \times \mathrm{SU}_3), & \mathrm{E}_6/(\mathrm{SU}_3 \times \mathrm{SU}_3 \times \mathrm{SU}_3), \\ \mathrm{E}_7/(\mathrm{SU}_3 \times \mathrm{SU}_6), & \mathrm{E}_8/\mathrm{SU}_9, & \mathrm{E}_8/(\mathrm{E}_6 \times \mathrm{SU}_3). & \end{array}$$

For invariant metric connections different from ∇^α, we prove that

Theorem A.2. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective non-symmetric SII space which admits at least one invariant metric connection, different from the Lie bracket family. Then, $M = G/K$ is isometric to a space given in Table 2. In this table we present the dimensions of the spaces $\mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m})$ and $(\Lambda^3\mathfrak{m})^K$, which respectively parametrize the space of invariant metric connections and the space of invariant metric connections with totally skew-symmetric torsion. In particular:*
(i) Any homogeneous space in Table 2 whose isotropy representation is of real type and which is not isometric to $\mathrm{SO}_{10}/\mathrm{Sp}_2$, admits a 2-dimensional space of G-invariant metric connections with skew-torsion. For $\mathrm{SO}_{10}/\mathrm{Sp}_2$, the unique family of SO_{10}-invariant metric connections with skew-torsion is given by ∇^α ($\alpha \in \mathbb{R}$). However, the space of all SO_{10}-invariant metric connections is 2-dimensional.
(ii) Any homogeneous space in Table 2 whose isotropy representation is of complex type, admits a 6-dimensional space of G-invariant metric connections and a 4-dimensional subspace of G-invariant metric connections with skew-torsion.

For invariant connections induced by some $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$, we prove the following:

Theorem B. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective non-symmetric SII space, which admits at least one invariant affine connection ∇^μ, induced by some $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$. Then:*
(i) If the associated isotropy representation is of real type, then $M = G/K$ is isometric to a manifold in Table 3. In this table, **s** *states for the dimension of the module $\mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$, which parametrizes the space of such invariant connections.*
(ii) If the associated isotropy representation is of complex type, then $M = G/K$ is isometric to one of the manifolds $\mathrm{SO}_{n^2-1}/\mathrm{SU}_n$ ($n \geq 4$) or $\mathrm{E}_6/\mathrm{SU}_3$, where the dimension of the space of such invariant connections is 2 and 4, respectively.
(iii) The G-invariant connection ∇^μ does not preserve the Killing metric $g = -B|_{\mathfrak{m}}$. Thus, ∇^μ is not metric with respect to any G-invariant metric.

Now, a small combination of Theorems A.1, A.2 and B yields the desired dimension of the space of all *G-invariant affine connections* for any non-symmetric (compact) SII space $M = G/K$,

$$\mathcal{N} := \dim_{\mathbb{R}} \mathit{Aff}_G(F(G/K)) = \dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}).$$

We refer to Tables 4 and 5, where the number $\mathcal{N}$ is explicitly indicated. Note that for SII homogeneous spaces $M = G/K$ of the Lie group $G = \mathrm{SU}_n$, we can describe explicitly some of the SU_n-invariant affine connections induced by a symmetric K-intertwining map $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$ (and in a few cases all such connections, see Corollary 4.8). We also conclude that the space of invariant *torsion-free connections* on a non-symmetric SII space $M = G/K$, denoted by $\mathit{Aff}^0_G(F(G/K))$,

2 The parameter α can be a real or complex number, depending on the type of the isotropy representation $\mathfrak{m}$.

3 http://wwwmathlabo.univ-poitiers.fr/~maavl/LiE/.

Table 2
Non-symmetric SII spaces carrying new G-invariant metric connections and the dimension of the space of global G-invariant 3-forms.

Real type			
$M = G/K$ (families)	$\dim_{\mathbb{R}} M$	Invariant metric connections $\dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m})$	skew-torsion $\dim_{\mathbb{R}}(\Lambda^3\mathfrak{m})^K$
$\mathrm{SU}_{pq}/\mathrm{SU}_p \times \mathrm{SU}_q$ $(p, q \geq 3)$	$(p^2-1)(q^2-1)$	2	2
$\mathrm{SO}_{\frac{n(n-1)}{2}}/\mathrm{SO}_n$ $(n \geq 9)$	$\frac{1}{8}(6n - 5n^2 - 2n^3 + n^4)$	3	2
$\mathrm{SO}_{\frac{(n-1)(n+2)}{2}}/\mathrm{SO}_n$ $(n \geq 7)$	$\frac{1}{8}(8 - 2n - 9n^2 + 2n^3 + n^4)$	3	2
$\mathrm{SO}_{(n-1)(2n+1)}/\mathrm{Sp}_n$ $(n \geq 4)$	$\frac{1}{2}(2 + n - 9n^2 - 4n^3 + 4n^4)$	3	2
$\mathrm{SO}_{n(2n+1)}/\mathrm{Sp}_n$ $(n \geq 3)$	$\frac{1}{2}(-3n - 5n^2 + 4n^3 + 4n^4)$	3	2
(low-dim cases)			
$\mathrm{SO}_{21}/\mathrm{SO}_7$	189	3	2
$\mathrm{SO}_{28}/\mathrm{SO}_8$	350	4	2
$\mathrm{SO}_{14}/\mathrm{SO}_5$	81	3	2
$\mathrm{SO}_{20}/\mathrm{SO}_6$	175	3	2
$\mathrm{SO}_{10}/\mathrm{Sp}_2$	35	2	1
(exceptions)			
$\mathrm{SO}_{14}/\mathrm{G}_2$	70	2	2
$\mathrm{SO}_{26}/\mathrm{F}_4$	273	2	2
$\mathrm{SO}_{42}/\mathrm{Sp}_4$	825	2	2
$\mathrm{SO}_{52}/\mathrm{F}_4$	1274	2	2
$\mathrm{SO}_{70}/\mathrm{SU}_8$	2352	2	2
$\mathrm{SO}_{248}/\mathrm{E}_8$	30 380	2	2
$\mathrm{SO}_{78}/\mathrm{E}_6$	2925	2	2
$\mathrm{SO}_{128}/\mathrm{Spin}_{16}$	8008	2	2
$\mathrm{SO}_{133}/\mathrm{E}_7$	8645	2	2
$\mathrm{E}_7/\mathrm{SU}_3$	125	2	2
Complex type			
$M = G/K$	$\dim_{\mathbb{R}} M$	Invariant metric connections $\dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m}, \Lambda^2(\mathfrak{m}))$	skew-torsion $\dim_{\mathbb{R}}(\Lambda^3\mathfrak{m})^K$
$\mathrm{SO}_{n^2-1}/\mathrm{SU}_n$ $(n \geq 4)$	$\frac{1}{2}(4 - 5n^2 + n^4)$	6	4
$\mathrm{E}_6/\mathrm{SU}_3$	70	6	4

Table 3
Non-symmetric SII spaces of real type carrying G-invariant affine connections induced by $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$.

Real type		
$\mathbf{s} = 1$	$\mathbf{s} = 2$	$\mathbf{s} = 3$
$\mathrm{SU}_{10}/\mathrm{SU}_5$	$\mathrm{SU}_{\frac{n(n-1)}{2}}/\mathrm{SU}_n$ $(n \geq 6)$	$\mathrm{SO}_{28}/\mathrm{SO}_8$
$\mathrm{SO}_{\frac{n(n-1)}{2}}/\mathrm{SO}_n$ $(n \geq 9)$	$\mathrm{SU}_{\frac{n(n+1)}{2}}/\mathrm{SU}_n$ $(n \geq 3)$	$\mathrm{E}_7/\mathrm{SU}_3$
$\mathrm{SO}_{\frac{(n-1)(n+2)}{2}}/\mathrm{SO}_n$ $(n \geq 7)$	$\mathrm{SO}_{20}/\mathrm{SO}_6$	
$\mathrm{SO}_{21}/\mathrm{SO}_7$	$\mathrm{SU}_{27}/\mathrm{E}_6$	
$\mathrm{SO}_{14}/\mathrm{SO}_5$	$\mathrm{SU}_{pq}/\mathrm{SU}_p \times \mathrm{SU}_q$ $(p, q \geq 3)$	
$\mathrm{SO}_{(n-1)(2n+1)}/\mathrm{Sp}_n$ $(n \geq 4)$		
$\mathrm{SO}_{n(2n+1)}/\mathrm{Sp}_n$ $(n \geq 3)$		
$\mathrm{SU}_{2q}/\mathrm{SU}_2 \times \mathrm{SU}_q$ $(q \geq 3)$		
$\mathrm{SO}_{10}/\mathrm{Sp}_2$		
$\mathrm{SU}_{16}/\mathrm{Spin}_{10}$		
$\mathrm{SO}_{70}/\mathrm{SU}_8$		
$\mathrm{E}_6/\mathrm{G}_2$		
$\mathrm{E}_6/(\mathrm{G}_2 \times \mathrm{SU}_3)$		

is parametrized by an affine subspace of $\mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, which is modelled on $\mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$ and contains the Levi-Civita connection, see Lemma 1.4 and Remark 1.6. In particular, for any G-invariant affine connection ∇^μ induced by $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$, the invariant connection $\nabla := \nabla^\mu - \frac{1}{2}T^\mu$ is torsion-free. Thus, the following is a direct consequence of Theorem B.

Corollary of Theorem B. *The classification of non-symmetric SII spaces which admit new invariant torsion-free connections, in addition to the Levi-Civita connection, is given by the manifolds of Theorem B. In particular, for a space in Table 3 we have* $\dim_{\mathbb{R}} \mathcal{A}ff^0_G(F(G/K)) = \mathbf{s}$, *and for the almost complex homogeneous spaces in Theorem B it is* $\dim_{\mathbb{R}} \mathcal{A}ff^0_G(F(G/K)) = 2$ *or* 4, *respectively.*

Classification of ∇-Einstein structures with skew-torsion. After obtaining Theorems A.1, A.2 and B, in the final Section 5 we turn our attention to more geometric problems. We use our classification results of Table 2 to examine ∇-Einstein structures

Table 4
The multiplicities **a**, **s**, $\mathcal{N}$ and ℓ for (non-symmetric) SII homogeneous spaces—Classical families.

Classical families and their associated low-dimensional cases							
G	$M = G/K$	$\mathfrak{m}^{\mathbb{C}}$	**a**	**s**	$\mathcal{N}$	ℓ	Type
SU_n	(1) $\mathrm{SU}_{\frac{n(n-1)}{2}}/\mathrm{SU}_n$ $(n \geq 6)$	$R(\pi_2 + \pi_{n-2})$	1	2	3	1	r
	(1^*_α) $\mathrm{SU}_{10}/\mathrm{SU}_5$	$R(\pi_2 + \pi_3)$	1	1	2	1	r
	(2) $\mathrm{SU}_{\frac{n(n+1)}{2}}/\mathrm{SU}_n$ $(n \geq 3)$	$R(2\pi_1 + 2\pi_{n-1})$	1	2	3	1	r
	(3) $\mathrm{SU}_{pq}/\mathrm{SU}_p \times \mathrm{SU}_q$ $(p, q \geq 3)$	$R(\pi_1 + \pi_{p-1})\hat{\otimes}R(\pi_1 + \pi_{q-1})$	2	2	4	2	r
	(3^*_α) $\mathrm{SU}_{2q}/\mathrm{SU}_2 \times \mathrm{SU}_q$ $(q \geq 3)$	$R(2\pi_1)\hat{\otimes}R(\pi_1 + \pi_{q-1})$	1	1	2	1	r
SO_n	(4) $\mathrm{SO}_{n^2-1}/\mathrm{SU}_n$ $(n \geq 4)$	$R(2\pi_1 + \pi_{n-2}) \oplus R(\pi_2 + 2\pi_{n-1})$	6	2	8	4	c
	(4_α) $\mathrm{SO}_8/\mathrm{SU}_3$	$R(3\pi_1) \oplus R(3\pi_2)$	2	0	2	2	c
	(5) $\mathrm{SO}_{\frac{n(n-1)}{2}}/\mathrm{SO}_n$ $(n \geq 9)$	$R(\pi_1 + \pi_3)$	3	1	4	2	r
	(5_α) $\mathrm{SO}_{21}/\mathrm{SO}_7$	$R(\pi_1 + 2\pi_3)$	3	1	4	2	r
	(5_β) $\mathrm{SO}_{28}/\mathrm{SO}_8$	$R(\pi_1 + \pi_3 + \pi_4)$	4	3	7	2	r
	(6) $\mathrm{SO}_{\frac{(n-1)(n+2)}{2}}/\mathrm{SO}_n$ $(n \geq 7)$	$R(2\pi_1 + \pi_2)$	3	1	4	2	r
	(6_α) $\mathrm{SO}_{14}/\mathrm{SO}_5$	$R(2\pi_1 + 2\pi_2)$	3	1	4	2	r
	(6^*_β) $\mathrm{SO}_{20}/\mathrm{SO}_6$	$R(2\pi_1 + \pi_2 + \pi_3)$	3	2	5	2	r
	(7) $\mathrm{SO}_{(n-1)(2n+1)}/\mathrm{Sp}_n$ $(n \geq 4)$	$R(\pi_1 + \pi_3)$	3	1	4	2	r
	(7_α) $\mathrm{SO}_{14}/\mathrm{Sp}_3$	$R(\pi_1 + \pi_3)$	1	0	1	1	r
	(8) $\mathrm{SO}_{n(2n+1)}/\mathrm{Sp}_n$ $(n \geq 3)$	$R(2\pi_1 + \pi_2)$	3	1	4	2	r
	(8_α) $\mathrm{SO}_{10}/\mathrm{Sp}_2$	$R(2\pi_1 + \pi_2)$	2	1	3	1	r
	(9^*) $\mathrm{SO}_{4n}/\mathrm{Sp}_n \times \mathrm{Sp}_1$ $(n \geq 2)$	$R(\pi_2)\hat{\otimes}R(2\pi_1)$	1	0	1	1	r
Sp_n	(10) $\mathrm{Sp}_n/\mathrm{SO}_n \times \mathrm{Sp}_1$ $(n \geq 5)$	$R(2\pi_1)\hat{\otimes}R(2\pi_1)$	1	0	1	1	r
	(10_α) $\mathrm{Sp}_3/\mathrm{SO}_3 \times \mathrm{Sp}_1$	$R(4\pi_1)\hat{\otimes}R(2\pi_1)$	1	0	1	1	r
	(10_β) $\mathrm{Sp}_4/\mathrm{SO}_4 \times \mathrm{Sp}_1$	$R(2\pi_1 + 2\pi_2)\hat{\otimes}R(2\pi_1)$	1	0	1	1	r

with skew-torsion. Roughly speaking, such a structure consists of a n-dimensional connected Riemannian manifold (M, g) endowed with a metric connection ∇ which has non-trivial skew-torsion $0 \neq T \in \Lambda^3(T^*M)$ and whose Ricci tensor has symmetric part a multiple of the metric tensor, i.e. (see [25,5,3,17,18,22])

$$\mathrm{Ric}^{\nabla}_S = \frac{\mathrm{Scal}^{\nabla}}{n} g.$$

For $T = 0$ the whole notion reduces to the original Einstein metrics. In fact, like Einstein metrics on compact Riemannian manifolds, in [3] it was shown that ∇-Einstein structures can be characterized variationally. On the other hand, the classification of ∇-Einstein structures with skew-torsion on a fixed Riemannian manifold (M, g), is initially based on the classification of all metric connections on M whose torsion is a non-trivial 3-form. For example, for odd dimensional spheres $\mathrm{S}^{2n+1} \cong \mathrm{SU}_{n+1}/\mathrm{SU}_n$ endowed with their Sasakian structure, a classification of SU_{n+1}-*invariant* ∇-*Einstein structures with skew-torsion* has been very recently given in [22], and it follows only after the classification of SU_{n+1}-invariant metric connections (with skew-torsion) and their description in terms of tensor fields related to the special structure (see also [3]).

As far as we know, most well-understood examples of ∇-Einstein manifolds appear in the context of non-integrable geometries, where a metric connection with skew-torsion $0 \neq T$ is adapted to the geometry under consideration, the so-called *characteristic connection* ∇^c (see [25]). This connection, which in the homogeneous case coincides with the canonical connection, plays a crucial role in the theory of special geometries and nowadays is a traditional approach to describing the associated non-integrable structure in terms of ∇^c (or the very closely related *intrinsic torsion*). Moreover, the articles [25,5,3] provide some nice classes of ∇^c-Einstein structures, e.g. nearly Kähler manifolds in dimension 6, nearly-parallel G_2-manifolds in dimension 7, or 7-dimensional 3-Sasakian manifolds. Notice that these special structures admit (∇^c-parallel) real Killing spinors and hence, in some cases one can describe a deeper relation between the ∇-Einstein condition and a class of spinor fields, known as *Killing spinors with torsion*. These are natural generalizations of the original Killing spinor fields, satisfying the Killing spinor equation with respect to a metric connection with skew-torsion. Their existence is known for several types of special geometries (see [2,11,18]). For example, on 6-dimensional nearly Kähler manifolds, 7-dimensional nearly parallel G_2-manifolds, or even on $\mathrm{S}^3 \cong \mathrm{SU}_2$, such spinors are induced by the associated ∇^c-parallel spinors and their description is given in terms of whole 1-parameter families $\{\nabla^s : s \in \mathbb{R}\}$ of metric connections with skew-torsion. Moreover, their existence imposes the following strong geometric constraint: $\mathrm{Ric}^s = \frac{1}{n}\mathrm{Scal}^s g$ for any $s \in \mathbb{R}$ [18] (although in general this is not the case, see [11]). The special value $s = 1/4$ corresponds to the characteristic connection (which has parallel torsion T), while the parameter $s = 0$ induces the original Einstein metric related with the existent real Killing spinor.

Beside these classes of ∇-Einstein manifolds, the first author in [17] studies homogeneous ∇-Einstein structures for more general manifolds, e.g. on compact isotropy irreducible spaces and a class of normal homogeneous manifolds with two isotropy summands. An important result for us from [17], is that any effective compact isotropy irreducible homogeneous space $M = G/K$ which is not a symmetric space of Type I, is a ∇^α-Einstein manifold for any parameter $\alpha \neq 0$, where ∇^α is the Lie bracket family. As a consequence of the results in Section 4, we conclude that any (effective) non-symmetric SII homogeneous space $(M = G/K, g = -B|_{\mathfrak{m}})$ is a ∇^α-Einstein manifold for any parameter $\alpha \neq 0$. Moreover, our Lemma 3.12

Table 5
The multiplicities **a**, **s**, $\mathcal{N}$ and ℓ for (non-symmetric) SII homogeneous spaces—Exceptions.

Exceptions (“exceptions” in terms of [13, p. 203])

G	$M = G/K$	$\mathfrak{m}^{\mathbb{C}}$	**a**	**s**	$\mathcal{N}$	ℓ	Type
SU_n	$\mathrm{SU}_{16}/\mathrm{Spin}_{10}$	$R(\pi_4+\pi_5)$	1	1	2	1	r
	$\mathrm{SU}_{27}/\mathrm{E}_6$	$R(\pi_1+\pi_6)$	1	2	3	1	r
SO_n	$\mathrm{SO}_7/\mathrm{G}_2$	$R(\pi_1)$	1	0	1	1	r
	$\mathrm{SO}_{14}/\mathrm{G}_2$	$R(3\pi_1)$	2	0	2	2	r
	$\mathrm{SO}_{16}/\mathrm{Spin}_9$	$R(\pi_3)$	1	0	1	1	r
	$\mathrm{SO}_{26}/\mathrm{F}_4$	$R(\pi_3)$	2	0	2	2	r
	$\mathrm{SO}_{42}/\mathrm{Sp}_4$	$R(2\pi_3)$	2	0	2	2	r
	$\mathrm{SO}_{52}/\mathrm{F}_4$	$R(\pi_2)$	2	0	2	2	r
	$\mathrm{SO}_{70}/\mathrm{SU}_8$	$R(\pi_3+\pi_5)$	2	1	3	2	r
	$\mathrm{SO}_{248}/\mathrm{E}_8$	$R(\pi_7)$	2	0	2	2	r
	$\mathrm{SO}_{78}/\mathrm{E}_6$	$R(\pi_4)$	2	0	2	2	r
	$\mathrm{SO}_{128}/\mathrm{Spin}_{16}$	$R(\pi_6)$	2	0	2	2	r
	$\mathrm{SO}_{133}/\mathrm{E}_7$	$R(\pi_3)$	2	0	2	2	r
Sp_n	$\mathrm{Sp}_2/\mathrm{SU}_2$	$R(6\pi_1)$	1	0	1	1	r
	$\mathrm{Sp}_7/\mathrm{Sp}_3$	$R(2\pi_3)$	1	0	1	1	r
	$\mathrm{Sp}_{10}/\mathrm{SU}_6$	$R(2\pi_3)$	1	0	1	1	r
	$\mathrm{Sp}_{16}/\mathrm{Spin}_{12}$	$R(2\pi_6)$ or $R(2\pi_5)$	1	0	1	1	r
	$\mathrm{Sp}_{28}/\mathrm{E}_7$	$R(2\pi_7)$	1	0	1	1	r
G_2	$\mathrm{G}_2/\mathrm{SU}_3$	$R(\pi_1)\oplus R(\pi_2)$	2	0	2	2	c
	$\mathrm{G}_2/\mathrm{SO}_3$	$R(10\pi_1)$	1	0	1	1	r
F_4	$\mathrm{F}_4/(\mathrm{SU}_3^1\times\mathrm{SU}_3^2)$	$(R(2\pi_1)\hat{\otimes}R(\omega_1))\oplus(R(2\pi_2)\hat{\otimes}R(\omega_2))$	2	0	2	2	c
	$\mathrm{F}_4/(\mathrm{G}_2\times\mathrm{SU}_2)$	$R(\pi_1)\hat{\otimes}R(4\omega_1)$	1	0	1	1	r
E_6	$\mathrm{E}_6/\mathrm{SU}_3$	$R(4\pi_1+\pi_2)\oplus R(\pi_1+4\pi_2)$	6	4	10	4	c
	$\mathrm{E}_6/(\mathrm{SU}_3\times\mathrm{SU}_3\times\mathrm{SU}_3)$	$(R(\pi_1)\hat{\otimes}R(\omega_1)\hat{\otimes}R(\theta_1))\oplus$ $(R(\pi_2)\hat{\otimes}R(\omega_2)\hat{\otimes}R(\theta_2))$	2	0	2	2	c
	$\mathrm{E}_6/\mathrm{G}_2$	$R(\pi_1+\pi_2)$	1	1	2	1	r
	$\mathrm{E}_6/(\mathrm{G}_2\times\mathrm{SU}_3)$	$R(\pi_1)\hat{\otimes}R(\omega_1+\omega_2)$	1	1	2	1	r
E_7	$\mathrm{E}_7/\mathrm{SU}_3$	$R(4\pi_1+4\pi_2)$	2	3	5	2	r
	$\mathrm{E}_7/(\mathrm{SU}_3\times\mathrm{SU}_6)$	$(R(\pi_1)\hat{\otimes}R(\omega_2))\oplus(R(\pi_2)\hat{\otimes}R(\omega_5))$	2	0	2	2	c
	$\mathrm{E}_7/(\mathrm{G}_2\times\mathrm{Sp}_3)$	$R(\pi_1)\hat{\otimes}R(\omega_2)$	1	0	1	1	r
	$\mathrm{E}_7/(\mathrm{F}_4\times\mathrm{SU}_2)$	$R(\pi_4)\hat{\otimes}R(2\omega_1)$	1	0	1	1	r
E_8	$\mathrm{E}_8/\mathrm{SU}_9$	$R(\pi_3)\oplus R(\pi_6)$	2	0	2	2	c
	$\mathrm{E}_8/(\mathrm{F}_4\times\mathrm{G}_2)$	$R(\pi_4)\hat{\otimes}R(\omega_1)$	1	0	1	1	r
	$\mathrm{E}_8/(\mathrm{E}_6\times\mathrm{SU}_3)$	$(R(\pi_1)\hat{\otimes}R(\omega_1))\oplus(R(\pi_6)\hat{\otimes}R(\omega_2))$	2	0	2	2	c

in combination with Schur's lemma, yield a natural parameterization of the set of G-invariant ∇-Einstein structures with skew-torsion, by the space of invariant metric connections with non-trivial skew-torsion, or equivalently of the vector space of (global) invariant 3-forms. Hence, in this case the space of all homogeneous ∇-Einstein structures with skew-torsion on $(M = G/K, g = -B|_{\mathfrak{m}})$, denoted by $\mathcal{E}^{sk}_G(\mathrm{SO}(G/K, -B|_{\mathfrak{m}}))$, can be viewed as an affine subspace of the space of all G-invariant metric connections. Combining with our classification results on G-invariant metric connections with skew-torsion (see Theorems A.1, A.2, Table 2), we finally deduce that

Theorem C. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective non-symmetric SII space and assume that the family ∇^{α} exhausts all G-invariant metric connections. Then, the associated space of G-invariant ∇-Einstein structures with skew-torsion has dimension either*

$$\dim_{\mathbb{R}} \mathcal{E}^{sk}_G(\mathrm{SO}(G/K, -B|_{\mathfrak{m}})) = 1, \quad or \quad \dim_{\mathbb{R}} \mathcal{E}^{sk}_G(\mathrm{SO}(G/K, -B|_{\mathfrak{m}})) = 2,$$

for spaces with isotropy representation of real or complex type, respectively, and the manifold is one of the manifolds of Theorem A.1 or $\mathrm{SO}_{10}/\mathrm{Sp}_2$.

For the new invariant metric connections with skew-torsion, different from the family ∇^{α}, an explicit description seems difficult (for dimensional reasons, see Table 2). However, we prove that

Theorem D. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective non-symmetric SII space of Table 2, whose isotropy representation χ is of real type and assume that M is not isometric to $\mathrm{SO}_{10}/\mathrm{Sp}_2$. Then, the Ricci tensor associated to the 1-parameter family of invariant metric connections with skew-torsion, orthogonal to the Lie bracket family ∇^{α}, is also symmetric. Moreover,*

$$\dim_{\mathbb{R}} \mathcal{E}^{sk}_G(\mathrm{SO}(G/K, -B|_{\mathfrak{m}})) = 2.$$

This result is based on Theorem A.2 (Table 2) and the fact $(\Lambda^2\mathfrak{m})^K = 0$ for real representations of real type. This means that the space of skew-symmetric 2-forms $\Lambda^2\mathfrak{m}$ associated to a space $M = G/K$ of Theorem D (or even for the space $\mathrm{SO}_{10}/\mathrm{Sp}_2$), does not *contain the trivial representation, hence there do not exist G-invariant 2-forms. Consequently, the co-differential of the torsion associated to any existent G-invariant affine metric connection on M must vanish and our assertion follows by Schur's lemma in combination with the expression of the Ricci tensor for a metric connection with skew-torsion.*

Theorems C and D give the complete classification of all existent G-homogeneous ∇-Einstein structures on any effective, non-symmetric, SII space $M = G/K$, except of the quotients $\mathrm{SO}_{n^2-1}/\mathrm{SU}_n$ $(n \geq 4)$ and $\mathrm{E}_6/\mathrm{SU}_3$. These are privileged manifolds with respect to Theorem A.2; the associated space of G-invariant metric connections with skew-torsion is 4-dimensional. Moreover, they both admit an invariant almost complex structure and hence $\Lambda^2(\mathfrak{m})$ contains a copy of the trivial representation $\mathbb{R}$ (Lemma 5.4), i.e. there exist G-invariant (global) 2-forms. However, since we are interested only on the symmetric part of Ric^{∇} and the isotropy representation is (strongly) irreducible, again a combination of the results of Theorem A.2 with Schur's lemma, yields that

Theorem E. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be one of the manifolds $\mathrm{SO}_{n^2-1}/\mathrm{SU}_n$ $(n \geq 4)$ or $\mathrm{E}_6/\mathrm{SU}_3$. Then, the space of G-homogeneous ∇-Einstein structures with skew-torsion has dimension*

$$\dim_{\mathbb{R}} \mathcal{E}^{sk}_G(\mathrm{SO}(G/K, -B|_{\mathfrak{m}})) = 4.$$

1. Preliminaries

1.1. Metric connections and their types

Consider a connected, oriented Riemannian manifold (M^n, g) and identify the tangent and cotangent bundle $TM \cong T^*M$ via the bundle isomorphism provided by the metric tensor g. Any metric connection $\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM) \cong \Gamma(TM \otimes TM)$ on M can be written as $\nabla_X Y = \nabla^g_X Y + A(X, Y)$ for any $X, Y \in \Gamma(TM)$, for some tensor $A \in TM \otimes \Lambda^2(TM)$, where ∇^g is the Levi-Civita connection. Let us denote by $A(X, Y, Z) := g(A(X, Y), Z)$ the induced tensor obtained by contraction with g. The affine connections on M which are compatible with g, form an affine space modelled on the sections of the tensor bundle

$$\mathcal{A} := \{A \in \otimes^3 TM : A(X, Y, Z) + A(X, Z, Y) = 0\} \cong TM \otimes \Lambda^2(TM),$$

which has fibre dimension $n^2(n-1)/2$. It is well-known that $\mathcal{A}$ coincides with the space of torsion tensors

$$\mathcal{T} = \{A \in \otimes^3 TM : A(X, Y, Z) + A(Y, X, Z) = 0\} \cong \Lambda^2(TM) \otimes TM.$$

Moreover, under the action of the structure group SO_n it decomposes into three irreducible representations $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$, defined by

$$\begin{aligned}
\mathcal{A}_1 &:= \{A \in \mathcal{A} : A(X, Y, Z) = g(X, Y)\varphi(Z) - g(X, Z)\varphi(Y),\ \varphi \in \Gamma(T^*M)\} \cong TM,\\
\mathcal{A}_2 &:= \{A \in \mathcal{A} : \mathfrak{S}^{X,Y,Z} A(X, Y, Z) = 0,\ \Phi(A) = 0\},\\
\mathcal{A}_3 &:= \{A \in \mathcal{A} : A(X, Y, Z) + A(Y, X, Z) = 0\} \cong \Lambda^3 TM.
\end{aligned}$$

Here, the map $\Phi : \mathcal{A} \to T^*M$ is given by $\Phi(A)(Z) := \mathrm{tr} A_Z := \sum_i A(e_i, e_i, Z)$, for a vector field $Z \in \Gamma(TM)$ and a (local) orthonormal frame $\{e_i\}$ of M. The torsion $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ of ∇ satisfies the relation $T(X, Y) = A(X, Y) - A(Y, X)$ and conversely, A is expressed in terms of T by the condition

$$2A(X, Y, Z) = T(X, Y, Z) - T(Y, Z, X) + T(Z, X, Y), \quad \forall X, Y, Z \in \Gamma(TM). \tag{1.1}$$

We say that ∇ is of *vectorial type* (and the same for its torsion) if $A \in \mathcal{A}_1 \cong TM$, of *Cartan type, or traceless cyclic* if $A \in \mathcal{A}_2$ and finally (totally) *skew-symmetric* (or, of skew-torsion) if $A \in \mathcal{A}_3 \cong \Lambda^3 TM$. Notice that for $n = 2$, $\mathcal{A} \cong \mathbb{R}^2$ is irreducible. For $n \geq 3$, the mixed types occur by taking the direct sums of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$:

$$\begin{aligned}
\mathcal{A}_1 \oplus \mathcal{A}_2 &= \{A \in \mathcal{A} : \mathfrak{S}^{X,Y,Z} A(X, Y, Z) = 0\},\\
\mathcal{A}_2 \oplus \mathcal{A}_3 &= \{A \in \mathcal{A} : \Phi(A) = 0\},\\
\mathcal{A}_1 \oplus \mathcal{A}_3 &= \{A \in \mathcal{A} : A(X, Y, Z) + A(Y, X, Z) = 2g(X, Y)\varphi(Z) - g(X, Z)\varphi(Y)\\
&\qquad -g(Y, Z)\varphi(X),\ \varphi \in \Gamma(T^*M)\}.
\end{aligned}$$

Usually, connections of type $\mathcal{A}_1 \oplus \mathcal{A}_2$ are called *cyclic* and connections of type $\mathcal{A}_2 \oplus \mathcal{A}_3$ are known as *traceless* connections.

Let us finally recall that a tensor field $A \in \mathcal{A}$ satisfying $\nabla A = 0 = \nabla R$, where R denotes the curvature of the metric connection $\nabla = \nabla^g + A$ is called a *homogeneous structure*. The existence of a metric connection with these properties implies that (M, g) is locally homogeneous and if in addition (M, g) is complete, then it is locally isometric to a homogeneous Riemannian manifold. In particular, a complete, connected and simply-connected Riemannian manifold (M, g) endowed with a metric connection ∇ solving the equations $\nabla A = 0 = \nabla R$ is a homogeneous Riemannian manifold, see [43] for more details and proofs.

1.2. Connections with skew-torsion and ∇-Einstein manifolds

Let (M^n, g) be a connected Riemannian manifold carrying a metric connection ∇ with skew-torsion $0 \neq T \in \Lambda^3(TM)$, i.e.

$$g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2}T(X, Y, Z).$$

We normalize the length of T such that $\|T\|^2 := (1/6)\sum_{i,j} g(T(e_i, e_j), T(e_i, e_j))$ and we denote by $\delta^\nabla T = -\sum_{i=1}^n e_i \lrcorner \nabla_{e_i} T$ the co-differential of T. It is easy to check that $\delta^g T = \delta^\nabla T$. It is also known that (see for example [28,21,25])

Lemma 1.1. *The Ricci tensor associated to ∇ is given by*

$$\mathrm{Ric}^\nabla(X, Y) \equiv \mathrm{Ric}(X, Y) = \mathrm{Ric}^g(X, Y) - \frac{1}{4}\sum_{i=1}^n g(T(e_i, X), T(e_i, Y)) - \frac{1}{2}(\delta^g T)(X, Y).$$

Thus, in contrast to the Riemannian Ricci tensor Ric^g, the Ricci tensor of ∇ is not symmetric; it decomposes into a symmetric and anti-symmetric part $\mathrm{Ric} = \mathrm{Ric}_S + \mathrm{Ric}_A$, given by

$$\mathrm{Ric}_S(X, Y) := \mathrm{Ric}^g(X, Y) - \frac{1}{4}S(X, Y), \quad \mathrm{Ric}_A(X, Y) := -\frac{1}{2}(\delta^g T)(X, Y),$$

respectively, where S is the symmetric tensor defined by $S(X, Y) = \sum_{i=1}^n g(T(e_i, X), T(e_i, Y))$.

Definition 1.2 (*[3]*)**.** A triple (M, g, T) is called a ∇-*Einstein manifold with non-trivial skew-torsion* $0 \neq T \in \Lambda^3(TM)$, or for short, a ∇-*Einstein manifold*, if the symmetric part Ric_S of the Ricci tensor associated to the metric connection $\nabla = \nabla^g + \frac{1}{2}T$ satisfies the equation

$$\mathrm{Ric}_S = \frac{\mathrm{Scal}}{n} g, \tag{1.2}$$

where $\mathrm{Scal} \equiv \mathrm{Scal}^\nabla$ is the scalar curvature associated to ∇ and $n = \dim_{\mathbb{R}} M$. If $\nabla T = 0$, then (M, g, T) is called a ∇-*Einstein manifold with parallel skew-torsion*.

Notice that in contrast to the Riemannian case, for a ∇-Einstein manifold the scalar curvature $\mathrm{Scal}^\nabla \equiv \mathrm{Scal} = \mathrm{Scal}^g - \frac{3}{2}\|T\|^2$ is not necessarily constant (for details see [3]). For parallel torsion, i.e. $\nabla T = 0$, one has $\delta^\nabla T = 0$ and the Ricci tensor becomes symmetric $\mathrm{Ric} = \mathrm{Ric}_S$. If in addition $\delta\,\mathrm{Ric}^g = 0$, then the scalar curvature is constant, similarly to an Einstein manifold. This is the case for any ∇-Einstein manifold (M, g, ∇, T) with *parallel skew-torsion* [3, Prop. 2.7].

1.3. Invariant connections

Consider a Lie group G acting transitively on a smooth manifold M and let us denote by $\pi : P \to M$ a G-homogeneous principal bundle over M with structure group U. Let K be the isotropy subgroup at the point $o = \pi(p_0) \in M$ with $p_0 \in P$ (this is a closed subgroup $K \subset G$). Then, there is a Lie group homomorphism $\lambda : K \to U$ and hence an action of K on U, given by $ku = \lambda(k)u$. This induces a G-homogeneous principal U-bundle $P_\lambda \to M = G/K$, defined by $P_\lambda := G \times_K U = G \times_\lambda U = G \times U/\sim$, where $(g, u) \sim (gk, \lambda(k^{-1})u)$ for any $g \in G, u \in U, k \in K$. Because the left action of G on P restricts to a left action of K on the fibre P_o of P over a base point $o = eK \in G/K$, for the original bundle P we have $P \cong G \times_K P_o$. But fixing a point $u_0 \in P_o$ we see that the map $U \to P_o$, $u \mapsto u_0 u$ is a diffeomorphism and hence we identify $P \cong G \times_K P_o = G \times_K U = P_\lambda$, see also [16,32].

For G-homogeneous principal U-bundles $P \cong P_\lambda \to G/K$, it makes sense to speak about G-invariant connections, i.e. connections for which the horizontal subspaces $\mathcal{H}_p$ are also invariant by the left G-action, $(L_g)_*\mathcal{H}_p = \mathcal{H}_{gp}$ for any $g \in G$ and $p \in P$. In other words, a connection in P_λ is G-invariant if and only if the associated connection form $Z \in \Omega^1(P, \mathfrak{u})$ is such that $(\tau'_g)^*Z = Z$, for all $g \in G$, where $\tau'_g : P \to P$ is the (right) U-equivariant bundle map.

Theorem 1.3 (*[44]*)**.** *Let $P \cong P_\lambda \to G/K$ be a G-homogeneous principal U-bundle associated to a homomorphism $\lambda : K \to U$, as above. Then, G-invariant connections on P_λ are in a bijective correspondence with linear mappings $\Lambda : \mathfrak{g} \to \mathfrak{u}$ satisfying the following conditions:*

(a) $\Lambda(X) = \lambda_*(X)$, for all $X \in \mathfrak{k} = T_eK$, where $\lambda_* : \mathfrak{k} \to \mathfrak{u}$ is the differential of λ,
(b) $\Lambda(\mathrm{Ad}(k)X) = \mathrm{Ad}(\lambda(k))\Lambda(X)$, for all $X \in \mathfrak{g} = T_eG$, $k \in K$.

1.4. Reductive homogeneous spaces

Consider now a reductive homogeneous space $M = G/K$, i.e. we assume that there is an orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of $\mathfrak{g} = T_eG$ with $\mathrm{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$. Then we may identify $\mathfrak{m} = T_oM$ at $o = eK \in M$ and the isotropy representation $\chi : K \to \mathrm{Aut}(\mathfrak{m})$ of K with the restriction of the adjoint representation $\mathrm{Ad}|_K$ on $\mathfrak{m}$. Therefore, there is a direct sum decomposition $\mathrm{Ad}\big|_K = \mathrm{Ad}_K \oplus \chi$ where Ad_K is the adjoint representation of K (see [10]). As a further consequence, we identify the tangent bundle TM and the frame bundle $F(M)$ of $M = G/K$ with the homogeneous vector bundle $G \times_K \mathfrak{m}$ and the homogeneous principal bundle $G \times_K \mathrm{GL}(\mathfrak{m})$, respectively, the latter with structure group $\mathrm{GL}(\mathfrak{m}) = \mathrm{GL}_n \mathbb{R}$ ($n = \dim_{\mathbb{R}} \mathfrak{m} = \dim_{\mathbb{R}} M$).

An invariant affine connection on $M = G/K$ is a principal connection on $F(G/K)$ that is G-invariant. By Theorem 1.3 such an affine connection is described by a $\mathbb{R}$-linear map $\Lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ which is equivariant under the isotropy representation, i.e. $\Lambda(\mathrm{Ad}(k)X) = \mathrm{Ad}(k)\Lambda(X)\,\mathrm{Ad}(k)^{-1}$ for any $X \in \mathfrak{m}$ and $k \in K$. Let us denote by $\mathrm{Hom}_K(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$ the set of such linear maps. The assignment $\Lambda(X)Y = \eta(X, Y)$ provides an identification of $\mathrm{Hom}_K(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$ (and hence of the space of G-invariant affine connections on $M = G/K$) with the set of all $\mathrm{Ad}(K)$-equivariant bilinear maps $\eta : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$, i.e.

$$\eta(\mathrm{Ad}(k)X, \mathrm{Ad}(k)Y) = \mathrm{Ad}(k)\eta(X, Y), \tag{1.3}$$

for any $X, Y \in \mathfrak{m}$ and $k \in K$. Moreover, since any such map $\eta : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ induces a unique linear map $\tilde{\eta} : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ with $\tilde{\eta}(X \otimes Y) = \eta(X, Y)$, one may further identify (see [32, Thm. 5.1])

$$\mathcal{A}\mathit{ff}_G\big(F(G/K)\big) \cong \mathrm{Hom}_K(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m})) \cong \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}),$$

where in general $\mathcal{A}\mathit{ff}_G(P)$ denotes the affine space of G-invariant affine connections on a homogeneous principal bundle $P \to G/K$ over $M = G/K$ and $\mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ is the space of K-intertwining maps $\mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$. Usually we shall work with K connected and in this case we may identify $\mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}) = \mathrm{Hom}_{\mathfrak{k}}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$. Due to the orthogonal splitting $\mathfrak{m} \otimes \mathfrak{m} = \Lambda^2\mathfrak{m} \oplus \mathrm{Sym}^2 \mathfrak{m}$ we also remark that

$$\mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}) = \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m}) \oplus \mathrm{Hom}_K(\mathrm{Sym}^2, \mathfrak{m}). \tag{1.4}$$

The linear map $\Lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ is usually called *Nomizu map* or *connection map* (for details see [9,29]) and it satisfies the relation $\Lambda(X) = -(\nabla_X - L_X)_o$, where L_X is the Lie derivative with respect to X. Hence it encodes most of the properties of ∇; for example, the torsion $T \in \Lambda^2(\mathfrak{m}) \otimes \mathfrak{m}$ and curvature $R \in \Lambda^2(\mathfrak{m}) \otimes \mathfrak{k}$ of ∇ are given by:

$$T(X, Y)_o = \Lambda(X)Y - \Lambda(Y)X - [X, Y]_{\mathfrak{m}}, \quad R(X, Y)_o = [\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]_{\mathfrak{m}}) - \mathrm{ad}([X, Y]_{\mathfrak{k}}).$$

Lemma 1.4 (*[29]*). *Let $M = G/K$ be a homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Let $\Lambda, \Lambda' \in \mathrm{Hom}_{\mathfrak{k}}(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$ be two connection maps and let $\nabla, \nabla' \in \mathcal{A}\mathit{ff}_G\big(F(G/K)\big)$ be the associated G-invariant affine connections. Set $\eta := \Lambda - \Lambda'$. Then*
(i) ∇ and ∇' have the same geodesics if and only if $\eta \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m})$.
(ii) ∇ and ∇' have the same torsion if and only if $\eta \in \mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$.

Consider now a homogeneous Riemannian manifold $(M = G/K, g)$. In this case G can be considered as a closed subgroup of the full isometry group $\mathrm{Iso}(M, g)$, which implies that K and the Lie subgroup $\mathrm{Ad}(K) \subset \mathrm{Ad}(G)$ are compact subgroups. Hence, there is always a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with respect to some $\mathrm{Ad}(K)$-invariant inner product in the Lie algebra $\mathfrak{g}$. We shall denote by $\langle\ ,\ \rangle$ the $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$ induced by g. We equivariantly identify the K-modules $\mathfrak{so}(\mathfrak{m}, g) \equiv \mathfrak{so}(\mathfrak{m}) = \Lambda^2(\mathfrak{m})$ via the isomorphism $X \wedge Y \mapsto \langle X, \cdot\rangle Y - \langle Y, \cdot\rangle X$, for any $X, Y \in \mathfrak{m}$. Consider the $\mathrm{SO}(\mathfrak{m})$-principal bundle $\mathrm{SO}(G/K) \to G/K$ of $\langle\ ,\ \rangle$-orthonormal frames. This is a homogeneous principal bundle and an invariant metric connection on $M = G/K$ is a principal connection on $\mathrm{SO}(G/K)$ that is G-invariant. It follows that

Lemma 1.5. *A G-invariant affine connection ∇ on $(M = G/K, g)$ preserves the G-invariant Riemannian metric g if and only if the associated Nomizu map satisfies $\Lambda(X) \in \mathfrak{so}(\mathfrak{m}, g)$ for any $X \in \mathfrak{m}$.*

Notice that the existence of an invariant metric means that the isotropy representation of $M = G/K$ is self-dual, $\mathfrak{m} \simeq \mathfrak{m}^*$. Thus we may equivariantly identify

$$\mathfrak{gl}(\mathfrak{m}) \simeq \mathrm{End}(\mathfrak{m}) \simeq \mathfrak{m} \otimes \mathfrak{m}, \quad \mathrm{Hom}_K(\mathfrak{m}, \mathrm{End}(\mathfrak{m})) = (\mathfrak{m}^* \otimes \mathfrak{m}^* \otimes \mathfrak{m})^K \simeq (\otimes^3\mathfrak{m})^K.$$

In the last case, a K-equivariant map Λ on the left hand side is equivalent to a K-invariant tensor on the right hand side: $\mathrm{Hom}_K(\mathfrak{m}, \mathrm{End}(\mathfrak{m})) = (\otimes^3\mathfrak{m})^K$. The latter space has the following obvious K-submodules: $\Lambda^2\mathfrak{m} \otimes \mathfrak{m}$, $\mathrm{Sym}^2 \mathfrak{m} \otimes \mathfrak{m}$, $\mathfrak{m} \otimes \mathrm{Sym}^2 \mathfrak{m}$ and $\mathfrak{m} \otimes \Lambda^2\mathfrak{m}$. Of these, the last space corresponds to the $\mathfrak{so}(\mathfrak{m})$-valued Nomizu maps, i.e. the space of homogeneous metric connections which we denote by $\mathcal{M}_G(\mathrm{SO}(G/K))$. In particular, there is an equivariant isomorphism

$$\mathcal{M}_G(\mathrm{SO}(G/K, g)) \cong \mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m}).$$

Remark 1.6. The other submodules have different interpretations. For example, $\mathrm{Sym}^2 \mathfrak{m} \otimes \mathfrak{m}$ is the vector space on which the affine space of invariant torsion-free connections $\mathcal{A}\mathit{ff}^0_G(F(G/K))$ is modelled, and $\Lambda^2\mathfrak{m} \otimes \mathfrak{m}$ is the vector space on which the affine space of possible invariant torsion tensors is modelled. In fact, since the rearrangement of indices is equivariant (even with respect to the bigger algebra $\mathfrak{gl}(\mathfrak{m})$), one has the following isomorphisms: $\Lambda^2\mathfrak{m} \otimes \mathfrak{m} \simeq \mathfrak{m} \otimes \Lambda^2\mathfrak{m}$ and $\mathrm{Sym}^2 \mathfrak{m} \otimes \mathfrak{m} \simeq \mathfrak{m} \otimes \mathrm{Sym}^2 \mathfrak{m}$. Let us now relate this to the question of multiplicities of $\mathfrak{m}$ inside $\otimes^2\mathfrak{m} = \mathrm{End}(\mathfrak{m})$. Suppose we have a copy of $\mathfrak{m}$ inside the invariant decomposition of $\Lambda^2\mathfrak{m}$ (or respectively, in $\mathrm{Sym}^2 \mathfrak{m}$). This is equivalent to a map $\theta : \mathfrak{m} \to \Lambda^2\mathfrak{m}$ (respectively $\mathfrak{m} \to \mathrm{Sym}^2 \mathfrak{m}$). We may then raise all indices of θ to produce a K-invariant element of $\otimes^3\mathfrak{m}$. However through our freedom to rearrange indices, we may change to which of our four submodules this tensor belongs. For example, one may interpret the tensor corresponding to the instance of $\mathfrak{m}$ in $\Lambda^2\mathfrak{m}$ either as a metric connection in $\mathfrak{m} \otimes \Lambda^2\mathfrak{m}$, or a potentially non-metric connection in $\Lambda^2\mathfrak{m} \otimes \mathfrak{m}$. These coincide up to a scalar when $\theta \in \Lambda^3\mathfrak{m}$.

On a homogeneous Riemannian manifold $(M = G/K, g)$ the Levi-Civita connection ∇^g is the unique G-invariant metric connection determined by (cf. [13,35])

$$\langle \nabla^g_X Y, Z\rangle = -\frac{1}{2}\big[\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle [Y,Z]_{\mathfrak{m}}, X\rangle - \langle [Z,X]_{\mathfrak{m}}, Y\rangle\big], \quad \forall X, Y, Z \in \mathfrak{m}. \tag{$\star$}$$

On the other hand, the *canonical connection* on $M = G/K$ is induced by the principal K-bundle $G \to G/K$ and depends on the choice of the reductive complement $\mathfrak{m}$. It is defined by the horizontal distribution $\{\mathcal{H}_g := d\ell_g(\mathfrak{m}) : g \in G\}$, where ℓ_g denotes the left translation on G and its Nomizu map is given by $\Lambda^c : \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \xrightarrow{\mathrm{pr}_{\mathfrak{k}}} \mathfrak{k} \xrightarrow{\chi_*} \mathfrak{so}(\mathfrak{m})$, i.e. $\Lambda^c = \chi_* \circ \mathrm{pr}_{\mathfrak{k}}$. Thus, $\Lambda^c(X) = 0$ for any $X \in \mathfrak{m}$ (cf. [29,9]). Both the torsion $T^c(X,Y) = -[X,Y]_{\mathfrak{m}}$ and the curvature $R^c(X,Y) = -\mathrm{ad}([X,Y]_{\mathfrak{k}})$ of ∇^c are parallel objects, in particular any G-invariant tensor field on $M = G/K$ is ∇^c-parallel (cf. [36,29]). Hence, any homogeneous Riemannian manifold $(M = G/K, g)$ admits a homogeneous structure $A^c \in \mathfrak{m} \otimes \Lambda^2\mathfrak{m} \cong \mathcal{A}$ induced by the canonical connection ∇^c associated to the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. In the following, we shall refer to this homogeneous structure as the *canonical homogeneous structure*, adapted to $\mathfrak{m}$ and G. Using ($\star$) it is easy to see that $A^c := \nabla^c - \nabla^g$ satisfies the relation

$$A^c(X,Y,Z) = \frac{1}{2}T^c(X,Y,Z) - \langle U(X,Y), Z\rangle, \quad \forall X, Y, Z \in \mathfrak{m}, \tag{1.5}$$

where $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is the symmetric bilinear mapping defined by

$$2\langle U(X,Y), Z\rangle = \langle [Z,X]_{\mathfrak{m}}, Y\rangle + \langle X, [Z,Y]_{\mathfrak{m}}\rangle. \tag{1.6}$$

2. Invariant connections and derivations

Given a reductive homogeneous space $M = G/K$ endowed with a G-invariant affine connection ∇, in the following we examine $\mathrm{Ad}(K)$-equivariant derivations on $\mathfrak{m}$ induced by ∇ in terms of Nomizu maps. For the case of a compact Lie group G, this problem has been analysed in [17].

2.1. Derivations and generalized derivations

For the rest of this section let us fix a (connected) homogeneous manifold $M = G/K$ with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. For simplicity we assume that the transitive G-action is effective. We consider a bilinear mapping $\mu : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ and denote by $\Lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ the adjoint map, defined by $\Lambda(X)Y = \mu(X,Y)$.

Definition 2.1. The endomorphism $\Lambda(Z) : \mathfrak{m} \to \mathfrak{m}$ $(Z \in \mathfrak{m})$ is called a *derivation* of $\mathfrak{m}$, with respect to the Lie bracket operation $\mathrm{ad}_{\mathfrak{m}} := [\ ,\ ]_{\mathfrak{m}} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$, $\mathrm{ad}_{\mathfrak{m}}(X,Y) := [X,Y]_{\mathfrak{m}}$, if and only if $\mathfrak{der}^{\mu}(X,Y;Z) = 0$ identically, where for any $X, Y, Z \in \mathfrak{m}$ we set

$$\begin{aligned}\mathfrak{der}^{\mu}(X,Y;Z) &:= \Lambda(Z)[X,Y]_{\mathfrak{m}} - [\Lambda(Z)X, Y]_{\mathfrak{m}} - [X, \Lambda(Z)Y]_{\mathfrak{m}}\\ &= \mu(Z, [X,Y]_{\mathfrak{m}}) - [\mu(Z,X), Y]_{\mathfrak{m}} - [X, \mu(Z,Y)]_{\mathfrak{m}}.\end{aligned}$$

From now on, let us denote by $\mathrm{Der}(\mathrm{ad}_{\mathfrak{m}}; \mathfrak{m}) \equiv \mathrm{Der}(\mathfrak{m})$ the vector space of all derivations on $\mathfrak{m}$. We mention that given a bilinear map $\mu : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$, the condition $\mu \in \mathrm{Der}(\mathfrak{m})$ is equivalent to say that the associated connection map Λ is valued in $\mathrm{Der}(\mathfrak{m})$, i.e. $\Lambda \in \mathrm{Hom}(\mathfrak{m}, \mathrm{Der}(\mathfrak{m}))$. Restricting on K-intertwining maps $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ the vector space $\mathrm{Der}(\mathfrak{m})$ becomes a K-module, denoted by $\mathrm{Der}_K(\mathfrak{m})$. In fact, in this case we shall speak about $\mathrm{Ad}(K)$-equivariant derivations on $\mathfrak{m}$. So, let us focus on $\mathrm{Ad}(K)$-equivariant derivations induced by invariant connections on $M = G/K$.

Proposition 2.2. *Let $\nabla \equiv \nabla^{\mu}$ be a G-invariant connection on $M = G/K$ corresponding to $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$. Then, ∇^{μ} induces a $\mathrm{Ad}(K)$-equivariant derivation $\mu \in \mathrm{Der}_K(\mathfrak{m})$, if and only if $\mathrm{ad}_{\mathfrak{m}} := [\ ,\ ]_{\mathfrak{m}}$ is ∇^{μ}-parallel, i.e. $\nabla^{\mu}\,\mathrm{ad}_{\mathfrak{m}} = 0$ (which is equivalent to say that the torsion T^c of the canonical connection ∇^c associated to the reductive complement $\mathfrak{m}$ is ∇^{μ}-parallel, i.e. $\nabla^{\mu}T^c = 0$).*

Proof. The equivalence $\mu \in \mathrm{Der}(\mathrm{ad}_{\mathfrak{m}}; \mathfrak{m}) \equiv \mathrm{Der}(\mathfrak{m}) \Leftrightarrow \nabla^{\mu}\,\mathrm{ad}_{\mathfrak{m}} \equiv 0$ is an immediate consequence of the identity

$$\mathfrak{der}^{\mu}(X,Y;Z) = (\nabla^{\mu}_Z\,\mathrm{ad}_{\mathfrak{m}})(X,Y) = -(\nabla^{\mu}_Z T^c)(X,Y), \quad \forall X, Y, Z \in \mathfrak{m}. \tag{2.1}$$

The proof of (2.1) relies on the fact that G-invariant tensor fields are ∇^c-parallel, where ∇^c is the canonical connection associated to $\mathfrak{m}$. In particular, since ∇ is a G-invariant connection we write $\nabla^{\mu}_Z = \nabla^c_Z + \Lambda(Z)$, for any $Z \in \mathfrak{m}$, where $\Lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ is the associated Nomizu map. Then, for any $X, Y, Z \in \mathfrak{m}$ we obtain that

$$\begin{aligned}(\nabla^{\mu}_Z\,\mathrm{ad}_{\mathfrak{m}})(X,Y) &= \nabla^{\mu}_Z\,\mathrm{ad}_{\mathfrak{m}}(X,Y) - \mathrm{ad}_{\mathfrak{m}}(\nabla^{\mu}_Z X, Y) - \mathrm{ad}_{\mathfrak{m}}(X, \nabla^{\mu}_Z Y)\\ &= \big[\nabla^c_Z\,\mathrm{ad}_{\mathfrak{m}}(X,Y) - \mathrm{ad}_{\mathfrak{m}}(\nabla^c_Z X, Y) - \mathrm{ad}_{\mathfrak{m}}(X, \nabla^c_Z Y)\big]\\ &\quad + \big[\Lambda(Z)\,\mathrm{ad}_{\mathfrak{m}}(X,Y) - \mathrm{ad}_{\mathfrak{m}}(\Lambda(Z)X, Y) - \mathrm{ad}_{\mathfrak{m}}(X, \Lambda(Z)Y)\big]\\ &= (\nabla^c_Z\,\mathrm{ad}_{\mathfrak{m}})(X,Y) + \mathfrak{der}^{\mu}(X,Y;Z) = \mathfrak{der}^{\mu}(X,Y;Z),\end{aligned}$$

where the last equality follows since $\nabla^c\,\mathrm{ad}_{\mathfrak{m}} \equiv 0$. Similarly for the second equality in (2.1). $\square$

Example 2.3. The canonical connection ∇^c associated to the reductive complement $\mathfrak{m}$ induces a derivation on $\mathfrak{m}$ (the zero one, corresponding to $0 \in \mathrm{Der}_K(\mathfrak{m})$), since $\nabla^c T^c = 0$, or in other words since T^c is ∇^μ-parallel, where $\mu = 0 \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$.

Let us now generalize the notion of derivations on $\mathfrak{m}$, as follows:

Definition 2.4. Consider a tensor $F : \otimes^p \mathfrak{m} \to \mathfrak{m}$. Then, a bilinear mapping $\mu : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ is said to be a *generalized derivation* of F on $\mathfrak{m}$, if and only if μ satisfies the relation

$$\begin{aligned}\mu(Z, F(X_1, \ldots, X_p)) &= F(\mu(Z, X_1), X_2, \ldots, X_p) + \cdots + F(X_1, \ldots, X_{p-1}, \mu(Z, X_p)) \Leftrightarrow \\ \Lambda(Z)F(X_1, \ldots, X_p) &= F(\Lambda(Z)X_1, X_2, \ldots, X_p) + \cdots + F(X_1, \ldots, X_{p-1}, \Lambda(Z)X_p),\end{aligned}$$

for any $Z, X_1, \ldots, X_p \in \mathfrak{m}$, where $\Lambda \in \mathrm{Hom}(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$ is the adjoint map induced by μ.

For a tensor $F : \otimes^p \mathfrak{m} \to \mathfrak{m}$, the definition of a generalized derivation implies that if $\mu_1, \mu_2 : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ are two such bilinear mappings, then the linear combination $a\mu_1 + b\mu_2$ is also a generalized derivation of F on $\mathfrak{m}$. Hence, the set $\mathrm{Der}(F; \mathfrak{m})$ of all generalized derivations of F on $\mathfrak{m}$ is a vector space. Obviously, for $F = \mathrm{ad}_{\mathfrak{m}}$, a generalized derivation is just a classical derivation on $\mathfrak{m}$. Notice however that F can be much more general than the Lie bracket restriction, e.g. the torsion, or the curvature of a G-invariant connection ∇ on $M = G/K$ induced by some $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, or even μ itself. In particular, one may restrict Definition 2.4 on K-intertwining maps $\mu \in \mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m}, \mathfrak{m})$; then, the space $\mathrm{Der}(F; \mathfrak{m})$ becomes a K-module, which we shall denote by $\mathrm{Der}_K(F; \mathfrak{m})$. If moreover we focus on G-invariant tensor fields, then similarly to Proposition 2.2 we conclude that

Theorem 2.5. *Let $(M = G/K, \mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m})$ be a reductive homogeneous space endowed with an $\mathrm{Ad}(K)$-invariant tensor $F : \otimes^p \mathfrak{m} \to \mathfrak{m}$. Consider a K-intertwining map $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ and let us denote by ∇^μ the associated G-invariant affine connection. Then, μ is an $\mathrm{Ad}(K)$-equivariant generalized derivation of F if and only if F is ∇^μ-parallel, i.e. $\mu \in \mathrm{Der}_K(F; \mathfrak{m}) \Leftrightarrow \nabla^\mu F \equiv 0$.*

Proof. A direct computation shows that the evaluation of the covariant differentiation ∇F at the point $o = eK \in G/K$ gives rise to the following $\mathrm{Ad}(K)$-invariant tensor on $\mathfrak{m}$:

$$\begin{aligned}(\nabla_Z F)(X_1, \ldots, X_p) &= \nabla_Z F(X_1, \ldots, X_p) - \sum_{i=1}^{p} F(X_1, \ldots, \nabla_Z X_i, \ldots, X_p) \\ &= \nabla^c_Z F(X_1, \ldots, X_p) + \Lambda(Z)F(X_1, \ldots, X_p) - \sum_{i=1}^{p} F(X_1, \ldots, \nabla^c_Z X_i, \ldots, X_p) \\ &\quad - \sum_{i=1}^{p} F(X_1, \ldots, \Lambda(Z)X_i, \ldots, X_p) \\ &= \nabla^c_Z F(X_1, \ldots, X_p) - \sum_{i=1}^{p} F(X_1, \ldots, \nabla^c_Z X_i, \ldots, X_p) \\ &\quad + \Lambda(Z)F(X_1, \ldots, X_p) - \sum_{i=1}^{p} F(X_1, \ldots, \Lambda(Z)X_i, \ldots, X_p) \\ &= (\nabla^c_Z F)(X_1, \ldots, X_p) + (\mathcal{D}^\mu_Z F)(X_1, \ldots, X_p),\end{aligned}$$

where we set $(\mathcal{D}^\mu_Z F)(X_1, \ldots, X_p) := \Lambda(Z)F(X_1, \ldots, X_p) - \sum_{i=1}^{p} F(X_1, \ldots, \Lambda(Z)X_i, \ldots, X_p)$. However, F is by assumption G-invariant, hence $\nabla^c F = 0$ and our claim immediately follows. □

Moreover, we see that

Corollary 2.6. *On a reductive homogeneous space $(M = G/K, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m})$, given an $\mathrm{Ad}(K)$-invariant tensor $F : \otimes^p \mathfrak{m} \to \mathfrak{m}$ and some K-intertwining map $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, the operation*

$$(\mathcal{D}^\mu_Z F)(X_1, \ldots, X_p) := \Lambda(Z)F(X_1, \ldots, X_p) - \sum_{i=1}^{p} F(X_1, \ldots, \Lambda(Z)X_i, \ldots, X_p)$$

coincides with the covariant differentiation of F with respect to the connection $\nabla = \nabla^\mu$ induced on $M = G/K$ by μ, i.e. $(\nabla^\mu_Z F)(X_1, \ldots, X_p) = (\mathcal{D}^\mu_Z F)(X_1, \ldots, X_p)$ for any $X_1, \ldots, X_p, Z \in \mathfrak{m}$.

For a bilinear mapping $\mu : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ let us now introduce the tensor $\mathcal{C}^\mu$, defined by

$$\mathcal{C}^\mu(X, Y; Z) := (\nabla^\mu_Z \mu)(X, Y) - (\nabla^\mu_Z \mu)(Y, X),$$

for any $X, Y, Z \in \mathfrak{m}$. If $\mu \in \mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m}, \mathfrak{m})$, then we get the further identification $\mathcal{C}^\mu(X, Y; Z) := (\mathcal{D}^\mu_Z \mu)(X, Y) - (\mathcal{D}^\mu_Z \mu)(Y, X)$. In terms of $\mathcal{C}^\mu$ it is easy to check that

Proposition 2.7. *Let $\nabla = \nabla^{\mu}$ be a G-invariant affine connection on a reductive homogeneous space $(M = G/K, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m})$, corresponding to some $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$. Then, $\mu \in \mathrm{Der}_K(\mathfrak{m})$, if and only if*

$$(\nabla_Z T)(X, Y) \equiv (\mathcal{D}^{\mu}_Z T)(X, Y) = \mathcal{C}^{\mu}(X, Y; Z), \quad \forall X, Y, Z \in \mathfrak{m},$$

where $T = T^{\mu}$ is the torsion associated to ∇^{μ}.

Consequently, for some $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ the condition $\mu \in \mathrm{Der}_K(\mathfrak{m})$ can also be read in terms of the $\mathrm{Ad}(K)$-invariant tensor $\mathcal{C}^{\mu}$, which geometrically, represents the difference

$$(\nabla_Z T)(X, Y) - (\nabla_Z T^c)(X, Y) \equiv (\mathcal{D}^{\mu}_Z T)(X, Y) - (\mathcal{D}^{\mu}_Z T^c)(X, Y),$$

for any $X, Y, Z \in \mathfrak{m}$. In particular, a combination of Proposition 2.7 and identity (1.4), yields that

Theorem 2.8. *Let $M = G/K$ an effective homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then the following hold:*
(1) A G-invariant affine connection $\nabla = \nabla^{\mu}$ on $M = G/K$ corresponding to $\mu \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m})$, induces an $\mathrm{Ad}(K)$-equivariant derivation $\mu \in \mathrm{Der}_K(\mathfrak{m})$, if and only if

$$(\nabla^{\mu}_Z T)(X, Y) \equiv (\mathcal{D}^{\mu}_Z T)(X, Y) = 2\mathfrak{S}_{X,Y,Z}\,\mu(X, \mu(Y, Z)), \tag{2.2}$$

for any $X, Y, Z \in \mathfrak{m}$. This is equivalent to say that

$$(\nabla^{\mu}_Z T)(X, Y) \equiv (\mathcal{D}^{\mu}_Z T)(X, Y) = 2\big\{R(Z, X)Y + \Lambda(Y)(\Lambda(Z)X - [Z, X]_{\mathfrak{m}}) + \mathrm{ad}([Z, X]_{\mathfrak{k}})Y\big\}, \tag{2.3}$$

where R is the curvature tensor associated to ∇.
(2) A G-invariant affine connection $\nabla = \nabla^{\mu}$ on $M = G/K$ corresponding to $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$, induces an $\mathrm{Ad}(K)$-equivariant derivation $\mu \in \mathrm{Der}_K(\mathfrak{m})$ on $\mathfrak{m}$ if and only if the torsion T^{μ} associated to ∇^{μ} is ∇^{μ}-parallel.
(3) Let $\mu \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m})$. Then μ is an $\mathrm{Ad}(K)$-equivariant generalized derivation of itself, i.e. $\mu \in \mathrm{Der}_K(\mu; \mathfrak{m})$ if and only if $\mathcal{C}^{\mu} = 0$ identically.

Proof. For a skew-symmetric mapping $\mu \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m})$ a simple computation gives that

$$\mathcal{C}^{\mu}(X, Y; Z) = 2\mathfrak{S}_{X,Y,Z}\,\mu(X, \mu(Y, Z)).$$

Hence, (2.2) is an immediate consequence of Proposition 2.7. For the second relation (2.3), using the definition of the curvature tensor R and (2.7), for some $\mu \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m})$ we get that

$$\begin{aligned}(\nabla_Z T)(X, Y) = (\mathcal{D}^{\mu}_Z T)(X, Y) &= 2R(Z, X)Y + 2\Lambda(Y)(\Lambda(Z)X - [Z, X]_{\mathfrak{m}}) \\ &\quad + 2\,\mathrm{ad}([Z, X]_{\mathfrak{k}})Y - \mathfrak{der}^{\mu}(X, Y; Z),\end{aligned} \tag{2.4}$$

for any $X, Y, Z \in \mathfrak{m}$ and our claim immediately follows.

For the second statement and for a symmetric map $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$ it is easy to see that $\mathcal{C}^{\mu} = 0$. Therefore, our assertion is a direct consequence of Proposition 2.7.

Let us finally prove (3). By definition, it is $\mu \in \mathrm{Der}_K(\mu; \mathfrak{m})$ or equivalent $\Lambda \in \mathrm{Hom}_K(\mathfrak{m}, \mathrm{Der}_K(\mu; \mathfrak{m}))$, if and only if

$$\mu(Z, \mu(X, Y)) = \mu(\mu(Z, X), Y) + \mu(X, \mu(Z, Y))$$

for any $X, Y, Z \in \mathfrak{m}$, which is equivalent to say that $\mathfrak{S}_{X,Y,Z}\,\mu(X, \mu(Y, Z)) = 0$ identically. But since $\mathcal{C}^{\mu}(X, Y; Z) = 2\mathfrak{S}_{X,Y,Z}\,\mu(X, \mu(Y, Z))$, we conclude. □

Remark 2.9. For a compact connected Lie group $G \cong (G \times G)/\Delta G$ endowed with a bi-invariant affine connection ∇ corresponding to a skew-symmetric mapping $\mu \in \mathrm{Hom}_G(\Lambda^2\mathfrak{g}, \mathfrak{g})$, formula (2.3) has been described in [17, Prop. 2.4]. In particular, in this case relation (2.4) reduces to

$$(\nabla_Z T)(X, Y) = 2R(Z, X)Y + 2\Lambda(Y)(\Lambda(Z)X - [Z, X]_{\mathfrak{m}}) - \mathfrak{der}_{\mathfrak{g}}(X, Y; Z),$$

for any $X, Y, Z \in \mathfrak{g} = T_eG$, see also [17, Prop. 2.4].

Example 2.10. Let $M = G/K$ an effective homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. We consider the restricted Lie bracket $\mathrm{ad}_{\mathfrak{m}} := [-, -]_{\mathfrak{m}} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ and denote the associated Nomizu map just by $\Lambda_{\mathfrak{m}}$. Obviously, $\mathrm{ad}_{\mathfrak{m}}$ *induces a derivation on $\mathfrak{m}$ if and only if* $\mathrm{Jac}_{\mathfrak{m}} \equiv 0$, where $\mathrm{Jac}_{\mathfrak{m}} : \mathfrak{m} \times \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is the trilinear map defined by

$$\mathrm{Jac}_{\mathfrak{m}}(X, Y, Z) := \mathfrak{S}_{X,Y,Z}[X, [Y, Z]_{\mathfrak{m}}]_{\mathfrak{m}} = [X, [Y, Z]_{\mathfrak{m}}]_{\mathfrak{m}} + [Y, [Z, X]_{\mathfrak{m}}]_{\mathfrak{m}} + [Z, [X, Y]_{\mathfrak{m}}]_{\mathfrak{m}},$$

for any $X, Y, Z \in \mathfrak{m}$. The same conclusion follows from Theorem 2.8. Indeed, let us denote by $\nabla^{\mathfrak{m}}$ the G-invariant connection associated to $\mathrm{ad}_{\mathfrak{m}}$ and by $T^{\mathfrak{m}}$ and $R^{\mathfrak{m}}$ its torsion and curvature, respectively. It is $T^{\mathfrak{m}}(X, Y) = [X, Y]_{\mathfrak{m}}$ and

$$(\nabla^{\mathfrak{m}}_Z T^{\mathfrak{m}})(X, Y) = (\mathcal{D}^{\alpha_{\mathfrak{m}}}_Z T^{\mathfrak{m}})(X, Y) = \mathrm{Jac}_{\mathfrak{m}}(X, Y, Z),$$

for any $X, Y, Z \in \mathfrak{m}$. Moreover, $R^{\mathfrak{m}}(Z, X)Y = \mathrm{Jac}_{\mathfrak{m}}(X, Y, Z) - [[Z, X]_{\mathfrak{k}}, Y]$ and since $\Lambda_{\mathfrak{m}}(Z)X = [Z, X]_{\mathfrak{m}}$, an application of Theorem 2.8, (1), shows that $\Lambda_{\mathfrak{m}} \in \mathrm{Hom}_K(\mathfrak{m}, \mathrm{Der}(\mathfrak{m}))$ if and only if $\mathrm{Jac}_{\mathfrak{m}}(X, Y, Z) = 0$ for any $X, Y, Z \in \mathfrak{m}$. In fact, for $\mu = \mathrm{ad}_{\mathfrak{m}}$ it is $\mathcal{C}^{\mathrm{ad}_{\mathfrak{m}}}(X, Y; Z) = 2\,\mathrm{Jac}_{\mathfrak{m}}(X, Y, Z)$, hence the same results follows by relation (2.2). Finally, for the same assertion one can even apply Theorem 2.8, (3) for $\mu = \mathrm{ad}_{\mathfrak{m}}$.

Note that if $M = G/K$ is an effective symmetric space, then $\mathrm{Jac}_{\mathfrak{m}}$ is identically zero and $\mathrm{ad}_{\mathfrak{m}}$ is a derivation trivially. For example, any compact connected Lie group $M = G$ with a bi-invariant metric can be viewed as a symmetric space of the form $(G \times G)/\Delta G$. The Cartan decomposition is given by $\mathfrak{g} \oplus \mathfrak{g} = \Delta\mathfrak{g} \oplus \mathfrak{m}$, where $\Delta\mathfrak{g} := \{(X, X) \in \mathfrak{g} \oplus \mathfrak{g} : X \in \mathfrak{g}\} \cong \mathfrak{g}$ and $\mathfrak{m} := \{(X, -X) \in \mathfrak{g} \oplus \mathfrak{g} : X \in \mathfrak{g}\} \cong \mathfrak{g}$, respectively. Obviously, in this case the condition $\mathrm{Jac}_{\mathfrak{m}} \equiv 0$ is the Jacobi identity which leads to the well-known result that the adjoint representation $\Lambda_{\mathfrak{g}} = \mathrm{ad}_{\mathfrak{g}}$ is a derivation of $\mathfrak{g}$. In the following section we examine the condition $\mathrm{Jac}_{\mathfrak{m}} \equiv 0$ also on non-symmetric, compact, effective and simply-connected naturally reductive manifolds, see Corollary 3.6.

3. Invariant connections on naturally reductive manifolds

Next we describe a series of new results related to invariant connections (and their torsion type) on effective naturally reductive spaces. Note that all these results can be applied on an effective, non-symmetric (compact) SII homogeneous Riemannian manifold.

3.1. Naturally reductive spaces

A Riemannian manifold (M, g) is called *naturally reductive* if there exists a closed subgroup G of the isometry group $\mathrm{Iso}(M, g)$ which acts transitively on M with isotropy group K and which induces a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that the torsion of the canonical connection ∇^c associated to $\mathfrak{m}$, is a 3-form $T^c \in \Lambda^3(\mathfrak{m})$. This is equivalent to say that $U \equiv 0$ identically, where $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is the bilinear map defined by (1.6). Thus, an alternative way to define naturally reductive manifolds is as follows:

Definition 3.1. A naturally reductive manifold is a homogeneous Riemannian manifold $(M = G/K, g)$ with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that canonical homogeneous structure $A^c \in \mathfrak{m} \otimes \Lambda^2\mathfrak{m}$ adapted to $\mathfrak{m}$ and G, is totally skew-symmetric, i.e. $2A^c(X, Y, Z) = T^c(X, Y, Z)$ for any $X, Y, Z \in \mathfrak{m}$.

A special subclass of naturally reductive manifolds $M = G/K$ consists of the so-called *normal* homogeneous Riemannian manifolds. In this case there is an $\mathrm{Ad}(G)$-invariant inner product Q on $\mathfrak{g}$ such that $Q(\mathfrak{k}, \mathfrak{m}) = 0$, i.e. $\mathfrak{m} = \mathfrak{k}^{\perp}$ and $Q|_{\mathfrak{m}} = \langle\, ,\, \rangle$. Thus, a normal metric is defined by a positive definite bilinear form Q. Notice however that Q can be more general, see [30,13]. If $Q = -B$, where B denotes the Killing form of $\mathfrak{g}$, then the normal metric is called the *Killing* (or *standard*) metric; this is the case if the Lie group G is compact and semi-simple. We mention that in this paper whenever we refer to a naturally reductive space $(M = G/K, g, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m})$ we shall always assume that G acts effectively on M and that $\mathfrak{g}$ coincides with the ideal $\tilde{\mathfrak{g}} := \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$. On the level of Lie groups this condition means that G coincides with the transvection group of the associated canonical connection ∇^c. Note that any compact normal homogeneous space satisfies this condition, see [41].

3.2. Properties of invariant connections in the naturally reductive setting

We start with the following simple observation.

Lemma 3.2. *Let $(M = G/K, g)$ be an effective compact homogeneous Riemannian manifold with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ which is not a symmetric space of Type I. Then, there is always an instance of $\mathfrak{m}$ inside $\Lambda^2(\mathfrak{m})$, associated to the restriction of the Lie bracket operation on the reductive complement $\mathfrak{m}$. In particular, this specific copy gives rise to a G-invariant metric connection on M if and only if g is naturally reductive with respect to G and $\mathfrak{m}$.*

Proof. Since $M = G/K$ is not isometric to a symmetric space of Type I, the canonical connection ∇^c has non-trivial torsion $T^c(X, Y) = -[X, Y]_{\mathfrak{m}}$, which gives rise to a non-trivial $\mathrm{Ad}(K)$-equivariant skew-symmetric bilinear mapping $\mathrm{ad}_{\mathfrak{m}} : \Lambda^2\mathfrak{m} \to \mathfrak{m}$. The second statement is apparent due to the naturally reductive property. □

Remark 3.3. If $(M = G/K, g)$ is a Riemannian symmetric space of Type I, then G is a compact simple Lie group and its Killing form B gives rise to a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. Moreover, the restriction $\langle\, ,\, \rangle = -B|_{\mathfrak{m}}$ induces a G-invariant metric which is naturally reductive with respect to $\mathfrak{m}$. However, the K-module $\Lambda^2(\mathfrak{m})$ never contains a copy of $\mathfrak{m}$, see also [33]. This is in contrast to a Riemannian symmetric space $(M = G, g = \rho)$ of Type II endowed with a bi-invariant metric ρ, where one can always find a copy of $\mathfrak{g}$ inside $\Lambda^2(\mathfrak{g})$, see also Remark 3.14. Geometrically, this copy represents the existence of 1-parameter family of canonical connections on any compact simple Lie group G (cf. [29,6,17]). The same is true in the more general compact case (cf. [38]).

Lemma 3.4 (*[1,17]*). *Let $(M = G/K, g)$ be an effective naturally reductive manifold with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that $\mathfrak{g} = \tilde{\mathfrak{g}}$. Then,*
(i) A G-invariant metric connection ∇ on $(M = G/K, g)$ has totally skew-symmetric torsion $T \in \Lambda^3(\mathfrak{m})$ if and only if $\Lambda(Z)Z = 0$, for any $Z \in \mathfrak{m}$, where Λ is the associated Nomizu map.
(ii) There is a bijective correspondence between $\mathrm{Ad}(K)$-equivariant maps $\Lambda^\alpha : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$, defined by $\Lambda^\alpha(X)Y = \frac{1-\alpha}{2}[X, Y]_{\mathfrak{m}} = (1-\alpha)\Lambda^g(X)Y$ for any $X, Y \in \mathfrak{m}$, and G-invariant metric connections ∇^α with totally skew-symmetric torsion $T^\alpha \in \Lambda^3(\mathfrak{m})$, such that $T^\alpha = \alpha \cdot T^c$ for some parameter α, where T^c is the torsion of the canonical connection ∇^c associated to $\mathfrak{m}$ and $\Lambda^g : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ the Nomizu map associated to the Levi-Civita connection ∇^g.

Let us finally recall the following fundamental result by Olmos and Reggiani.

Theorem 3.5 (*[38, Thm. 1.2], [37, Thm. 2.1]*). *Let (M^n, g) be a simply-connected and irreducible Riemannian manifold that is not isometric to a sphere, nor to a projective space, nor to a compact simple Lie group with a bi-invariant metric or its symmetric dual. Then the canonical connection is unique, i.e. (M^n, g) admits at most one naturally reductive homogeneous structure.*

Combining the observations in Example 2.10 and the results of Lemma 3.4 and Theorem 3.5, in the compact and simply-connected case we obtain the following conclusion about derivations on $\mathfrak{m}$:

Corollary 3.6. *Let $(M = G/K, g)$ be an effective, compact and simply-connected naturally reductive manifold, irreducible as Riemannian manifold, endowed with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that $\mathfrak{g} = \tilde{\mathfrak{g}}$. Assume that $M = G/K$ is not isometric to a symmetric space of Type I, neither to a sphere or to a real projective space. Then, the bilinear mapping $\mathrm{ad}_{\mathfrak{m}} := [\ ,\ ]_{\mathfrak{m}}$ gives rise to $\mathrm{Ad}(K)$-equivariant derivation on $\mathfrak{m}$, if and only if M is isometric to a compact simple Lie group G endowed with a bi-invariant metric.*

Proof. A special version of Corollary 3.6 has been proved in [17, Lem. 4.5]. Here we improve this result. We omit some details and only present the main idea. Assume that $\mathrm{ad}_{\mathfrak{m}} \in \mathrm{Der}_K(\mathfrak{m})$, i.e. $\mathrm{Jac}_{\mathfrak{m}}(X, Y, Z) = 0$ for any $X, Y, Z \in \mathfrak{m}$ (see Example 2.10). Then, for the family ∇^α of Lemma 3.4, a small computation shows that $\nabla^\alpha T^\alpha = 0$ for any α, see for example [1]. On the other hand, one can easily see that $[X, [Y, Z]_{\mathfrak{m}}]_{\mathfrak{k}} + [Y, [Z, X]_{\mathfrak{m}}]_{\mathfrak{k}} + [Z, [X, Y]_{\mathfrak{m}}]_{\mathfrak{k}} = 0$, for any $X, Y, Z \in \mathfrak{m}$. Combining this identity with $\mathrm{Jac}_{\mathfrak{m}}(X, Y, Z) = 0$, a long computation certifies that $\nabla^\alpha R^\alpha = 0$ for any α, as well. But then, ∇^α is a 1-parameter family of canonical connections on $M = G/K$ (in the sense of the Ambrose–Singer Theorem) and Theorem 3.5 yields the result. □

Notation: Let $(M = G/K, g)$ be an effective compact naturally reductive Riemannian manifold with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} = \tilde{\mathfrak{g}}$. If $\chi : K \to \mathrm{Aut}(\mathfrak{m})$ is of real type, we shall denote by $\mathbf{s}$ and $\mathbf{a}$ the multiplicity of $\mathfrak{m}$ inside $\mathrm{Sym}^2(\mathfrak{m})$ and $\Lambda^2(\mathfrak{m})$, respectively (or twice the multiplicity of $\mathfrak{m}$ inside $\mathrm{Sym}^2(\mathfrak{m})$ and $\Lambda^2(\mathfrak{m})$, respectively, if $\chi : K \to \mathrm{Aut}(\mathfrak{m})$ is of complex type). We also set

$$\mathcal{N} = \mathbf{s} + \mathbf{a} := \dim_{\mathbb{R}} \mathcal{Aff}_G(F(G/K)) = \dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$$
$$\mathcal{N}_{\mathrm{mtr}} := \dim_{\mathbb{R}} \mathcal{M}_G(\mathrm{SO}(G/K)) = \dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m}, \Lambda^2(\mathfrak{m})) \leq \mathcal{N}.$$

Since K is compact, and we treat finite dimensional K-representations, we conclude that

Lemma 3.7. *The dimensions of modules $\mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m})$ and $\mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m})$ coincide,*

$$\dim_{\mathbb{R}} \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m}) = \dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m}),$$

or in other words $\mathbf{a} = \mathcal{N}_{\mathrm{mtr}}$.

Remark 3.8. Note that there exists compact Lie groups, e.g. $G = \mathrm{U}_n$, admitting skew-symmetric $\mathrm{Ad}(G)$-equivariant maps $\Lambda^2(\mathfrak{g}) \to \mathfrak{g}$ which do not induce bi-invariant metric connections with respect to the bi-invariant inner product $\langle X, Y \rangle = -\operatorname{tr} XY$ (see also the proof of Theorem 3.15). In fact, below we will show that Lemma 3.7 implies that [6, Lem. 3.1] or [17, Corol. 2.3, Thm. 2.9] are in general false. In particular, the corresponding statements hold only for compact simple Lie groups, but fail for general compact Lie groups.

Next, our aim is to clarify Remark 3.8. For simplicity, given an $\mathrm{Ad}(K)$-equivariant bilinear mapping $\mu : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ associated to a G-invariant connection ∇ on $(M = G/K, g)$ we shall use the same notation for the corresponding K-intertwining map $\mu : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ (and we shall identify them) and denote by $\hat{\mu}$ the contraction of μ with the $\mathrm{Ad}(K)$-invariant inner product $\langle\ ,\ \rangle : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ associated to g, i.e. $\hat{\mu}(X, Y, Z) := \langle \mu(X, Y), Z \rangle$, for any $X, Y, Z \in \mathfrak{m}$. Notice that $\hat{\mu}$ is an $\mathrm{Ad}(K)$-invariant tensor on $\mathfrak{m}$. Initially, it is useful to examine invariant connections related to some $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$. Then, the induced tensor $\hat{\mu} = \hat{\mu}^{\mathbf{s}}$ is such that $\hat{\mu}(X, Y, Z) = \hat{\mu}(Y, X, Z)$ for any $X, Y, Z \in \mathfrak{m} \cong T_o G/K$ and the corresponding Nomizu map $\Lambda := \Lambda^{\mathbf{s}} : \mathfrak{m} \to \mathrm{Sym}^2(\mathfrak{m})$ is also symmetric in the sense that $\Lambda(X)Y = \Lambda(Y)X$ (since $\mu(X, Y) = \mu(Y, X)$ for any $X, Y \in \mathfrak{m}$). Next we prove that when $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$ is non-trivial, then the induced connection cannot preserve the naturally reductive metric $\langle\ ,\ \rangle$. We start with the non-symmetric case.

Lemma 3.9. *Let $(M = G/K, g)$ be a connected, compact, non-symmetric, naturally reductive space of a compact Lie group G modulo a compact subgroup K. Assume that the transitive G-action is effective and let us denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} = \tilde{\mathfrak{g}}$ the associated reductive decomposition. If ∇ is an invariant metric connection induced by some $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$, then $\mu = 0$ and ∇ coincides with the canonical connection ∇^c associated to $\mathfrak{m}$.*

In fact, the non-existence of an invariant metric connection $\nabla^{\mathbf{s}}$ corresponding to a non-trivial element $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$ can be proved also as follows. By the condition $T^{\mathbf{s}}(X, Y) = -[X, Y]_{\mathfrak{m}}$ and since $\langle\ ,\ \rangle$ is naturally reductive with respect to G, one concludes that $\nabla^{\mathbf{s}}$ is an invariant connection with skew-torsion, which according to Lemma 3.4 is equivalent to say that $\Lambda^{\mathbf{s}}(X)X = 0$ for any $X \in \mathfrak{m}$. But then, it is also $\Lambda^{\mathbf{s}}(X+Y)(X+Y) = 0$ for any $X, Y \in \mathfrak{m}$, i.e. $\Lambda^{s}(X)Y = -\Lambda^{s}(Y)X$, which gives rise to a contradiction (since $\mu \neq 0$). Let us now explain also the compact symmetric case.

Lemma 3.10. *Given a connected Riemannian symmetric space $(M = G/K, g)$ of Type I (resp. $(M = (G\times G)/\Delta(G), \rho)$ of Type II, for some compact, connected, simple Lie group G with a bi-invariant metric ρ), then the unique G-invariant (resp. bi-invariant) metric connection which is induced by a symmetric $\mathrm{Ad}(K)$-equivariant mapping on $\mathfrak{m}$ (resp, symmetric $\mathrm{Ad}(G)$-equivariant mapping on $\mathfrak{g}$), is the torsion-free Levi-Civita connection.*

Proof. Consider first a symmetric space $(M = G/K, g)$ of Type I, endowed with a G-invariant affine connection ∇^{μ} associated to an element $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$. Then, $T^{\mu}(X, Y) = 0$ for any $X, Y \in \mathfrak{m}$, i.e. $\hat{\mu} \in \mathrm{Sym}^2 \mathfrak{m} \otimes \mathfrak{m}$. Hence, assuming in addition that ∇ is metric with respect to g, the fundamental theorem in Riemannian geometry implies the identification of ∇ with the unique torsion-free metric connection on $(M = G/K, g)$, i.e. the Levi-Civita connection, or the canonical connection associated to $\mathfrak{m}$. In particular, $\mu = 0$ is trivial. The same conclusions, related this time to bi-invariant metric connections corresponding to maps $\mu \in \mathrm{Hom}_G(\mathrm{Sym}^2 \mathfrak{g}, \mathfrak{g})$, hold for a compact, connected, simple Lie group $G \cong (G \times G)/\Delta G$, endowed with a bi-invariant metric ρ. $\square$

Remark 3.11. Laquer proved in [33] the existence of (irreducible) compact symmetric spaces $(M = G/K, g)$ which admit invariant affine connections induced by non-trivial elements $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$. And indeed, by [6] we know that these G-invariant connections are not metric with respect to $g = -B|_{\mathfrak{m}}$, as it should be according to Lemma 3.10. The same is true for compact simple Lie groups, such as SU_n, see [32,6].

Let us now consider invariant connections whose torsion is a 3-form. We show that on an effective, non-symmetric, compact, naturally reductive space $(M = G/K, g)$ the G-invariant metric connections whose torsion is a 3-form necessarily correspond to instances of the trivial representation inside the space $\Lambda^3\mathfrak{m}$, and conversely. In particular, the torsion form is a G-invariant 3-form. Let us denote by ℓ the multiplicity of the (real) trivial representation inside $\Lambda^3\mathfrak{m}$.

Lemma 3.12. *Let $(M = G/K, g)$ be a naturally reductive manifold as in Lemma 3.9. Then, the dimension of the affine space of G-invariant metric connections on M which have the same geodesics with the Levi-Civita connection ∇^g, i.e. $\Lambda(X)X = 0$, or equivalent whose torsion form T is a non-trivial G-invariant 3-form, is equal to ℓ. In particular,*

$$1 \leq \ell \leq \mathcal{N}_{\mathrm{mtr}} = \mathbf{a} \leq \mathcal{N}.$$

Proof. First notice that $1 \leq \ell \leq \mathcal{N}_{\mathrm{mtr}}$. This follows since the induced $\mathrm{Ad}(K)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ satisfies the naturally reductive property and hence the torsion of the canonical connection $T^c(X, Y, Z) = -\langle [X, Y]_{\mathfrak{m}}, Z\rangle \neq 0$ is a non-trivial G-invariant 3-form. Then, according to Lemma 3.4, the family $\nabla^{\alpha} = \nabla^c + \Lambda^{\alpha}$ induces a 1-parameter family of metric connections with skew-torsion $T^{\alpha} := \alpha T^c \neq 0$. Now, any instance of the trivial representation inside $\Lambda^3(\mathfrak{m})$ induces a G-invariant (global) 3-form on $M = G/K$, say $0 \neq T \in \Lambda^3(\mathfrak{m})^K$. If $\ell \geq 2$, then we can also assume that $T \neq T^{\alpha}$. But then, one can define a 1-parameter family of metric connections with skew-torsion, say $2sT$, given by $\nabla^s = \nabla^g + sT$. Obviously, this family is G-invariant and preserves the metric. On the other hand, if $M = G/K$ admits a G-invariant metric connection ∇ with skew-torsion T such that $T \neq T^{\alpha}$, then T must be a global G-invariant 3-form and hence it corresponds to a new copy of the trivial representation inside $\Lambda^3\mathfrak{m}$. $\square$

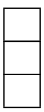
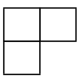
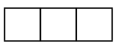
For a complete description of all G-invariant metric connections on $(M = G/K, g)$, one has to encode the "defect"

$$\epsilon := \mathcal{N}_{\mathrm{mtr}} - \ell \geq 0.$$

For this, it is useful to consider the tensor product

$$\otimes^3\mathfrak{m} = \mathfrak{m} \otimes \mathfrak{m} \otimes \mathfrak{m} \cong (\Lambda^2\mathfrak{m} \oplus \mathrm{Sym}^2 \mathfrak{m}) \otimes \mathfrak{m} \cong \big(\Lambda^2\mathfrak{m} \otimes \mathfrak{m}\big) \oplus \big(\mathrm{Sym}^2 \mathfrak{m} \otimes \mathfrak{m}\big)$$

and its decomposition in terms of Young diagrams:

$\otimes^3\mathfrak{m} =$  $\oplus\ 2$  $\oplus$ 

$$= \mathrm{Sym}^3 \mathfrak{m} \oplus \mathcal{L}(\mathfrak{m}) \oplus \Lambda^3(\mathfrak{m}),$$

where $\mathcal{L}(\mathfrak{m}) := \ker(P_{\mathbf{sym}}) \cap \ker(P_{\mathbf{skew}})$ is the section of the kernels of the equivariant projections

$$P_{\mathbf{sym}} : \otimes^3\mathfrak{m} \to \mathrm{Sym}^3(\mathfrak{m}), \quad P_{\mathbf{skew}} : \otimes^3\mathfrak{m} \to \Lambda^3(\mathfrak{m}).$$

Notice that the kernel of the natural maps $\mathrm{Sym}^2\mathfrak{m} \otimes \mathfrak{m} \to \mathrm{Sym}^3\mathfrak{m}$ and $\Lambda^2(\mathfrak{m}) \otimes \mathfrak{m} \to \Lambda^3(\mathfrak{m})$ are isomorphic irreducible $\mathrm{GL}(\mathfrak{m})$-modules of real dimension $n(n-1)(n+1)/3$, where $n := \dim_{\mathbb{R}}\mathfrak{m} = \dim_{\mathbb{R}} M$ (for an example see [42, p. 246]). Moreover, there is an equivariant isomorphism

$$\mathcal{L}(\mathfrak{m}) \cong \oplus^2 \ker(\mathfrak{m} \otimes \Lambda^2(\mathfrak{m}) \to \Lambda^3(\mathfrak{m})).$$

The intersection of $\mathcal{L}(\mathfrak{m})$ with the K-module $\mathfrak{m} \otimes \Lambda^2\mathfrak{m}$ consists of metric connections and is isomorphic to the so-called (2,1)-plethysm of the K-representation $\mathfrak{m}$:

$$P_{\mathfrak{m}}(2,1) := \mathcal{L}(\mathfrak{m}) \cap \big(\mathfrak{m} \otimes \Lambda^2\mathfrak{m}\big).$$

Theorem 3.13. *Let $(M^n = G/K, g = \langle\ ,\ \rangle, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m})$ as in Lemma 3.9. The existence of the trivial representation inside the $n(n-1)(n+1)/3$-dimensional (2, 1)-plethysm $P_{\mathfrak{m}}(2,1)$ of $\mathfrak{m}$, gives rise to a G-invariant metric connection $\nabla = \nabla^{\mu}$ on $M = G/K$ corresponding to a K-intertwining bilinear mapping $\mu : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ which has both non-trivial symmetric and skew-symmetric part, i.e. $\mu = \mu^{\mathrm{skew}} + \mu^{\mathrm{sym}}$, with $0 \neq \mu^{\mathrm{skew}} \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m})$ and $0 \neq \mu^{\mathrm{sym}} \in \mathrm{Hom}_K(\mathrm{Sym}^2\mathfrak{m}, \mathfrak{m})$, respectively. In particular, the torsion of ∇^{μ} is not totally skew-symmetric and the defect $\epsilon := \mathcal{N}_{\mathrm{mtr}} - \ell \geq 0$ coincides with the multiplicity of the trivial representation inside $P_{\mathfrak{m}}(2,1)$.*

Proof. The trivial representation inside $P_{\mathfrak{m}}(2,1)$ induces an $\mathrm{Ad}(K)$-equivariant 3-tensor $\hat{\mu}$ on $\mathfrak{m}$ which is skew-symmetric with respect the last two indices, i.e. $\hat{\mu} \in \mathfrak{m} \otimes \Lambda^2(\mathfrak{m})$. Since the K-module $\mathfrak{m} \otimes \Lambda^2\mathfrak{m}$ corresponds to the set of $\mathfrak{so}(\mathfrak{m})$-valued Nomizu maps on $M = G/K$ with respect to $\langle\ ,\ \rangle$, the induced invariant connection $\nabla = \nabla^{\mu}$ is necessarily metric. In order to prove that its torsion is not a 3-form we rely on the definition of $P_{\mathfrak{m}}(2,1)$ and the orthogonal decomposition

$$\otimes^3\mathfrak{m} = \mathrm{Sym}^3\mathfrak{m} \oplus \mathcal{L}(\mathfrak{m}) \oplus \Lambda^3(\mathfrak{m}). \tag{3.1}$$

Indeed, since the 3-tensor $\hat{\mu}(X, Y, Z) = \langle \mu(X, Y), Z\rangle$ is induced by the trivial representation inside $P_{\mathfrak{m}}(2,1) := \mathcal{L}(\mathfrak{m}) \cap \big(\mathfrak{m} \otimes \Lambda^2\mathfrak{m}\big)$, the direct sum decomposition (3.1) together with Lemma 3.12, shows that the torsion T^{μ} of ∇^{μ} cannot be totally skew-symmetric. We use now (1.4) and write $\mu = \mu^{\mathrm{skew}} + \mu^{\mathrm{sym}}$ for the corresponding K-intertwining bilinear mapping $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$. Since T^{μ} is not a 3-form, μ^{sym} cannot be trivial, $\mu^{\mathrm{sym}} \neq 0$. Indeed, if $\mu^{\mathrm{sym}} = 0$, then $\mu = \mu^{\mathrm{skew}}$ and hence $\mu(X, X) = 0$ for any $X \in \mathfrak{m}$. But then, using Lemma 3.4, (i) we get a contradiction. Assume now that μ is given by a (non-trivial) symmetric K-intertwining bilinear mapping, i.e $\mu^{\mathrm{skew}} = 0$ and $\mu = \mu^{\mathrm{sym}}$ where $0 \neq \mu^{\mathrm{sym}} : \mathrm{Sym}^2\mathfrak{m} \to \mathfrak{m}$. Then, according to Lemma 3.9 our connection ∇^{μ} cannot be metric with respect to $\langle\ ,\ \rangle$, which contradicts to $\hat{\mu} \in \mathfrak{m} \otimes \Lambda^2(\mathfrak{m})$. This shows that $\mu^{\mathrm{skew}} \neq 0$, as well. Now, the identification of the defect $\epsilon := \mathcal{N}_{\mathrm{mtr}} - \ell \geq 0$ with the multiplicity of the trivial representation in $P_{\mathfrak{m}}(2,1)$ is a direct consequence of (3.1) and Lemmas 3.7, 3.12. □

We mention that one cannot drop the naturally reductive assumption in Theorem 3.13, due to the fact that the proof relies on Lemmas 3.9 and 3.12.

Remark 3.14. On a compact simple Lie group G, bi-invariant connections which are compatible with the Killing form are induced by the copy of $\mathfrak{g}$ inside $\Lambda^2(\mathfrak{g})$. Indeed, recall that

$$\mathfrak{so}(\mathfrak{g}) \cong \Lambda^2(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^{\perp}, \quad \mathfrak{g}^{\perp} := \ker\delta_{\mathfrak{g}},$$

where $\delta_{\mathfrak{g}} : \Lambda^2(\mathfrak{g}) \to \mathfrak{g}$ is given by $\delta_{\mathfrak{g}}(X \wedge Y) := [X, Y]$. Since $\delta_{\mathfrak{g}}$ is surjective, $\mathfrak{g}$ always lies inside $\Lambda^2(\mathfrak{g})$. However, the module $P_{\mathfrak{g}}(2,1) := \mathcal{L}(\mathfrak{g}) \cap \big(\mathfrak{g} \otimes \Lambda^2\mathfrak{g}\big)$, where $\mathcal{L}(\mathfrak{g})$ is similarly defined by $\mathcal{L}(\mathfrak{g}) := \oplus^2 \ker\big(\mathfrak{g} \otimes \Lambda^2(\mathfrak{g}) \to \Lambda^3(\mathfrak{g})\big)$, never contains the trivial summand. In contrast, as we noticed in Remark 3.8, for a compact Lie group the case can be different. Let us focus for example on $G = \mathrm{U}_n$ ($n \geq 3$).

3.3. Bi-invariant metric connections on the compact Lie group U_n

According to Laquer [32], for $n \geq 3$ the space of bi-invariant affine connections on U_n is 6-dimensional. In particular, the following $\mathrm{Ad}(G)$-equivariant bilinear mappings form a basis of $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ for $\mathfrak{g} = \mathfrak{u}(n)$:

$$\left.\begin{array}{lll} \mu_1(X,Y) = [X,Y], & \mu_2(X,Y) = i(XY + YX), & \mu_3(X,Y) = i\,\mathrm{tr}(X)Y \\ \mu_4(X,Y) = i\,\mathrm{tr}(Y)X, & \mu_5(X,Y) = i\,\mathrm{tr}(XY)\,\mathrm{Id}, & \mu_6(X,Y) = i\,\mathrm{tr}(X)\,\mathrm{tr}(Y)\,\mathrm{Id} \end{array}\right\}, \tag{3.2}$$

where XY denotes multiplication of matrices and Id is the identity matrix. We also consider the linear combinations

$$\nu(X,Y) := \mu_3(X,Y) - \mu_4(X,Y) = i(\mathrm{tr}(X)Y - \mathrm{tr}(Y)X) \in \mathrm{Hom}_G(\Lambda^2\mathfrak{g}, \mathfrak{g}),$$
$$\vartheta(X,Y) := \mu_3(X,Y) + \mu_4(X,Y) = i(\mathrm{tr}(X)Y + \mathrm{tr}(Y)X) \in \mathrm{Hom}_G(\mathrm{Sym}^2\mathfrak{g}, \mathfrak{g}).$$

Theorem 3.15. *(1) The connection induced by the* $\mathrm{Ad}(\mathfrak{u}(n))$*-equivariant bilinear mapping* $\mu = \mu_4 - \mu_5$*, i.e.* $\mu(X, Y) := i(\mathrm{tr}(Y)X - \mathrm{tr}(XY)\mathrm{Id})$ *for any* $X, Y \in \mathfrak{u}(n)$*, is a bi-invariant metric connection on* U_n $(n \geq 3)$ *with respect to the bi-invariant metric induced by* $\langle\ ,\ \rangle = -\mathrm{tr}(XY)$*. The symmetric and skew-symmetric part of* $\mu = \mu^{\mathrm{skew}} + \mu^{\mathrm{sym}}$ *are given by*

$$\mu^{\mathrm{skew}}(X, Y) = -(1/2)\nu(X, Y), \quad \text{and} \quad \mu^{\mathrm{sym}}(X, Y) = (1/2)\vartheta(X, Y) + i\langle X, Y\rangle\,\mathrm{Id},$$

respectively, and its torsion has the form $T^{\mu}(X, Y) = -\nu(X, Y) + T^{c}(X, Y)$*. In particular, the induced tensor* $T^{\mu}(X, Y, Z) = \langle T^{\mu}(X, Y), Z\rangle$ *is not totally skew-symmetric.*
(2) Consequently, for $n \geq 3$ *the Lie group* U_n *carries a 2-dimensional space of bi-invariant metric connections, i.e.* $\mathcal{N}_{\mathrm{mtr}} = \epsilon + \ell = 2$*.*

Proof. (1) The module $\mathcal{L}(\mathfrak{g})$ associated to the adjoint representation of $\mathfrak{g} = \mathfrak{u}(n) = \mathbb{R} \oplus \mathfrak{su}(n)$ contains the trivial representation twice. The one copy corresponds to the invariant 3-tensor $\hat{\nu}(X, Y, Z) = \langle \nu(X, Y), Z\rangle$ which is skew-symmetric only with respect to the first two indices, i.e. $\hat{\nu} \in \mathcal{L}(\mathfrak{g}) \cap (\Lambda^2\mathfrak{g} \otimes \mathfrak{g})$ and thus $\nu = \mu_3 - \mu_4$ fails to induce a bi-invariant connection on U_n, preserving $\langle\ ,\ \rangle$. The second copy corresponds to the invariant 3-tensor $\hat{\mu}(X, Y, Z) = \langle \mu(X, Y), Z\rangle$, where $\mu : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is given by $\mu = \mu_4 - \mu_5$. We will show that $\hat{\mu}$ is indeed inside the $(2, 1)$ plethysm $P_{\mathfrak{g}}(2, 1) = \mathcal{L}(\mathfrak{g}) \cap (\mathfrak{g} \otimes \Lambda^2\mathfrak{g})$, i.e. $\epsilon = 1$, and hence the associated connection ∇^{μ} gives rise to 1-dimensional family of bi-invariant metric connections on U_n. For simplicity, we set $\mathcal{O}(X, Y, Z) := \langle \mu(X, Y), Z\rangle + \langle Y, \mu(X, Z)\rangle$, for any $X, Y, Z \in \mathfrak{u}(n)$. Then we get that

$$\begin{aligned}
\mathcal{O}(X, Y, Z) &= \langle i(\mathrm{tr}(Y)X - \mathrm{tr}(XY)\mathrm{Id}), Z\rangle + \langle Y, i(\mathrm{tr}(Z)X - \mathrm{tr}(XZ)\mathrm{Id})\rangle \\
&= i\big(\mathrm{tr}(Y)\langle X, Z\rangle - \mathrm{tr}(XY)\langle \mathrm{Id}, Z\rangle + \mathrm{tr}(Z)\langle Y, X\rangle - \mathrm{tr}(XZ)\langle Y, \mathrm{Id}\rangle\big) \\
&= i\big(-\mathrm{tr}(Y)\mathrm{tr}(XZ) + \mathrm{tr}(XY)\mathrm{tr}(Z) - \mathrm{tr}(Z)\mathrm{tr}(XY) + \mathrm{tr}(XZ)\mathrm{tr}(Y)\big) = 0,
\end{aligned}$$

for any $X, Y, Z \in \mathfrak{u}(n)$ and this proves our assertion. Now, according to Theorem 3.13, μ has both non-trivial symmetric and skew-symmetric part, namely $\mu^{\mathrm{sym}}(X, Y) = \frac{1}{2}[\mu(X, Y) + \mu(Y, X)]$ and $\mu^{\mathrm{skew}}(X, Y) = \frac{1}{2}[\mu(X, Y) - \mu(Y, X)]$, respectively, and a small computation completes the proof.
(2) For the second statement, the mapping $a\mu_1(X, Y)$ $(a \in \mathbb{R})$ induces a 1-parameter family of bi-invariant metric connections on U_n with skew-torsion and this is the unique family of bi-invariant metric connections with skew-torsion (since the multiplicity of the trivial representation inside $\Lambda^3\mathfrak{g}$ is one, i.e. $\ell = 1$, see also the remark below). Then, according to Theorem 3.13 it must be $\mathcal{N}_{\mathrm{mtr}} = \epsilon + \ell = 1 + 1 = 2$, which also fits with the conclusion that μ is a new bi-invariant metric connection on U_n, and finally also with Lemma 3.7. This induces a 1-parameter family of bi-invariant metric connections ∇^{b} $(b \in \mathbb{R})$, corresponding to the bilinear mapping $\mu^{b}(X, Y) := b[i(\mathrm{tr}(Y)X - \mathrm{tr}(XY)\mathrm{Id})] = b\mu(X, Y)$, with $X, Y \in \mathfrak{u}(n)$. The torsion $T^{b} \in \Lambda^2\mathfrak{u}(n) \otimes \mathfrak{u}(n)$ is not totally skew-symmetric. Indeed, the torsion of the mapping $\mu = \mu_4 - \mu_5$ is given by $T^{\mu}(X, Y) = -\nu(X, Y) + T^{c}(X, Y)$. It is not totally skew-symmetric since for example $\mu(X, X) \neq 0$ and $\langle\ ,\ \rangle$ is a naturally reductive metric. Similarly for μ^{b}. This finishes the proof. □

Remark 3.16. For a verification of the fact $\ell = 1$ for U_n, one can use the LiE program (and stability arguments), or even apply the following. First, for dimensional reasons notice that

$$\Lambda^3(\mathfrak{g}) = \Lambda^3(\mathfrak{u}(n)) = \Lambda^3(\mathbb{R} \oplus \mathfrak{su}(n)) = \Lambda^3(\mathfrak{su}(n)) \oplus \big(\mathbb{R} \otimes \Lambda^2\mathfrak{su}(n)\big). \tag{$*$}$$

Using (4.3) we also see that $\mathbb{R} \otimes \Lambda^2\mathfrak{su}(n)$ does not contain the trivial representation. For the decomposition of $\Lambda^3(\mathfrak{su}(n))$, recall first that any compact simple lie group $\hat{G}$ admits a non-trivial global $\hat{G}$-invariant 3-from, the so-called Cartan 3-form $\omega_{\hat{\mathfrak{g}}}(X, Y, Z) = B([X, Y], Z)$, where B denotes the Killing form on the Lie algebra $\hat{\mathfrak{g}}$. On the other hand, the $\mathrm{Ad}(\hat{G})$-equivariant differential $d_{\hat{\mathfrak{g}}} : \Lambda^k(\hat{\mathfrak{g}}) \to \Lambda^{k+1}(\hat{\mathfrak{g}})$ on $\hat{\mathfrak{g}}$ is defined by $d_{\hat{\mathfrak{g}}}(\psi \wedge \varphi) = d_{\hat{\mathfrak{g}}}(\psi) \wedge \varphi + (-1)^{\deg\psi}\psi \wedge d_{\hat{\mathfrak{g}}}(\varphi)$ with $d_{\hat{\mathfrak{g}}}(\varphi) = \sum_i (Z_i \lrcorner \omega_{\hat{\mathfrak{g}}}) \wedge (Z_i \lrcorner \varphi)$ for some $(-B)$-orthonormal basis $\{Z_i\}$ of $\hat{\mathfrak{g}}$. In these terms, in [34] it was shown that the splitting $\Lambda^3(\hat{\mathfrak{g}}) = \mathrm{span}_{\mathbb{R}}\{\omega_{\hat{\mathfrak{g}}}\} \oplus \delta_{\hat{\mathfrak{g}}}(\Lambda^4(\mathfrak{g})) \oplus d_{\hat{\mathfrak{g}}}(\hat{\mathfrak{g}}^{\perp})$ defines an equivariant orthogonal decomposition of $\Lambda^3(\hat{\mathfrak{g}})$, where $\delta_{\hat{\mathfrak{g}}}$ is the adjoint operator of $d_{\hat{\mathfrak{g}}}$ with respect to $-B$ (see also Remark 3.14). From this decomposition, one deduces that $\ell = 1$ for any compact simple Lie group $\hat{G}$, and since $\mathfrak{u}(n) = \mathbb{R} \oplus \hat{\mathfrak{g}}$ with $\hat{\mathfrak{g}} = \mathfrak{su}(n)$, by $(*)$ we conclude the same for U_n.

We finally observe that $\mu := \mu_4 - \mu_5$ does not induces a derivation on $\mathfrak{m}$ (apply for example Proposition 2.2 or Theorem 2.8). In particular, [17, Thm. 2.9] holds only for G compact and simple (the direct claim is true even in the compact case, but the converse direction fails for non-simple Lie groups, since [6, Lem. 3.1], or [17, Thm. 2.1], is valid only for a compact simple Lie group).

3.4. Characterization of the types of invariant metric connections

Given an effective naturally reductive Riemannian manifold $(M = G/K, g)$, our aim now is to characterize the possible invariant connections with respect to their torsion type (for skew-torsion, see [1] or Lemma 3.4). We remark that next is not necessary to assume the compactness of $M = G/K$.

Proposition 3.17. *Let* $(M^n = G/K, g)$ *be a homogeneous Riemannian manifold which is naturally reductive with respect to a closed subgroup* $G \subseteq \mathrm{Iso}(M, g)$ *of the isometry group and let* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ *be the associated reductive decomposition. Assume that the transitive G-action is effective,* $\mathfrak{g} = \tilde{\mathfrak{g}}$ *and denote by* $\nabla \equiv \nabla^{\mu}$ *a G-invariant metric connection corresponding to*

$\mu \in \mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m}, \mathfrak{m})$. Set $\hat{\mu}(X, Y, Z) = \langle \mu(X, Y), Z\rangle$, $A(X, Y) = \nabla_X Y - \nabla^g_X Y$ and $A(X, Y, Z) = \langle A(X, Y), Z\rangle$ for any $X, Y, Z \in \mathfrak{m}$, where ∇^g is the Levi-Civita connection. Then, the following hold:
(1) ∇ is of vectorial type, i.e. $A \in \mathcal{A}_1$, if and only if there is a global G-invariant 1-form φ on M such that

$$\hat{\mu}(X, Y, Z) = \frac{1}{2}\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle X, Y\rangle\varphi(Z) - \langle X, Z\rangle\varphi(Y), \quad \forall\, X, Y, Z \in \mathfrak{m}.$$

(2) ∇ is of Cartan type or traceless cyclic, i.e. $A \in \mathcal{A}_2$, if any only if the following two conditions are simultaneously satisfied:

$$\begin{array}{ll} (\alpha) & \mathfrak{S}_{X,Y,Z}\hat{\mu}(X, Y, Z) = \frac{3}{2}\langle [X, Y]_{\mathfrak{m}}, Z\rangle, \quad \forall\, X, Y, Z \in \mathfrak{m}, \\ (\beta) & \sum_i \mu(Z_i, Z_i) = 0, \end{array}$$

where $Z_1, \dots, Z_n$ is an arbitrary $\langle\ ,\ \rangle$-orthonormal basis of $\mathfrak{m}$.
(3) ∇ is cyclic, i.e. $A \in \mathcal{A}_1 \oplus \mathcal{A}_2$, if and only if $\mathfrak{S}_{X,Y,Z}\hat{\mu}(X, Y, Z) = \frac{3}{2}\langle [X, Y]_{\mathfrak{m}}, Z\rangle$, $\forall\, X, Y, Z \in \mathfrak{m}$.
(4) ∇ is traceless, i.e. $A \in \mathcal{A}_2 \oplus \mathcal{A}_3$, if and only if $\sum_i \mu(Z_i, Z_i) = 0$.

Remark 3.18. Before proceeding with the proof, let us first describe a useful formula. Recall that the torsion of ∇ is given by $T(X, Y) = \mu(X, Y) - \mu(Y, X) - [X, Y]_{\mathfrak{m}}$, or in other words $T(X, Y, Z) = \hat{\mu}(X, Y, Z) - \hat{\mu}(Y, X, Z) - \langle [X, Y]_{\mathfrak{m}}, Z\rangle$ for any $X, Y, Z \in \mathfrak{m}$. Therefore, a short application of (1.1) gives rise to

$$\begin{aligned} 2A(X, Y, Z) &= T(X, Y, Z) - T(Y, Z, X) + T(Z, X, Y) \\ &= \hat{\mu}(X, Y, Z) - \hat{\mu}(Y, X, Z) - \langle [X, Y]_{\mathfrak{m}}, Z\rangle - \hat{\mu}(Y, Z, Z) + \hat{\mu}(Z, Y, X) + \langle [Y, Z]_{\mathfrak{m}}, X\rangle \\ &\quad + \hat{\mu}(Z, X, Y) - \hat{\mu}(X, Z, Y) - \langle [Z, X]_{\mathfrak{m}}, Y\rangle \\ &= 2\hat{\mu}(X, Y, Z) - \langle [X, Y]_{\mathfrak{m}}, Z\rangle, \end{aligned}$$

since $\hat{\mu}(X, Y, Z) + \hat{\mu}(X, Z, Y) = 0$ for any $X, Y, Z \in \mathfrak{m}$ and $\langle\ ,\ \rangle$ is naturally reductive. Thus

$$A(X, Y, Z) = \hat{\mu}(X, Y, Z) - \frac{1}{2}\langle [X, Y]_{\mathfrak{m}}, Z\rangle, \quad \forall\, X, Y, Z \in \mathfrak{m}. \tag{3.3}$$

Proof. (1) Assume that $M = G/K$ carries a G-invariant metric connection ∇ whose torsion is of vectorial type. Next we shall identify $\mathfrak{m} \cong T_oM$ and for any $X \in \mathfrak{m} \subset \mathfrak{g}$ we shall write X^* for the (Killing) vector field on M induced by $\exp(-tX)$. Recall that $[X^*, Y^*]_o = -[X, Y]^*_o = -[X, Y]_{\mathfrak{m}}$. Since ∇ is a G-invariant connection, identifying $(\nabla_{X^*}Y^*)_o = \nabla_X Y$, we can write

$$\begin{aligned} \langle \nabla_X Y, Z\rangle &= \langle \nabla^c_X Y, Z\rangle + \langle \mu(X, Y), Z\rangle = \langle \nabla^c_X Y, Z\rangle + \langle \Lambda^{\mu}(X)Y, Z\rangle \\ &= -\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \hat{\mu}(X, Y, Z), \end{aligned} \tag{$*$}$$

where $(\nabla^c_{X^*}Y^*)_o = \nabla^c_X Y = -[X, Y]_{\mathfrak{m}} = [X^*, Y^*]_o$ is the canonical connection with respect to $\mathfrak{m}$ (cf. [38,41]). However, ∇ is of vectorial type, hence there is a 1-form φ on $M = G/K$ such that

$$\mathfrak{m} \otimes \Lambda^2\mathfrak{m} \cong \mathcal{A} \ni A(X, Y, Z) = \langle X, Y\rangle\varphi(Z) - \langle X, Z\rangle\varphi(Y),$$

for any $X, Y, Z \in \mathfrak{m}$. Using that $\langle\ ,\ \rangle$ is naturally reductive with respect to G and $\mathfrak{m}$, we compute $(\nabla^g_{X^*}Y^*)_o = \frac{1}{2}[X^*, Y^*]_o = -\frac{1}{2}[X, Y]_{\mathfrak{m}}$ and

$$\langle \nabla_X Y, Z\rangle = \langle \nabla^g_X Y, Z\rangle + A(X, Y, Z) = -\frac{1}{2}\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle X, Y\rangle\varphi(Z) - \langle X, Z\rangle\varphi(Y).$$

Hence, a small combination with ($*$) gives rise to

$$\hat{\mu}(X, Y, Z) = \frac{1}{2}\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle X, Y\rangle\varphi(Z) - \langle X, Z\rangle\varphi(Y). \tag{$**$}$$

However, $\hat{\mu}$ is an $\mathrm{Ad}(K)$-invariant tensor (or in other words, it corresponds to a G-invariant tensor field on $M = G/K$), and hence by ($**$) we conclude that φ must be a (global) G-invariant 1-form on M. This proves the one direction. Assume now that $(M = G/K, g)$ is endowed with a G-invariant tensor $\hat{\mu} \in \mathfrak{m} \otimes \Lambda^2\mathfrak{m}$ satisfying ($**$) for some G-invariant 1-form φ on M and let us denote by ∇ the associated G-invariant metric connection. Then, a combination of (3.3) and ($**$) yields that $A \in \mathcal{A}_1$, which completes the proof of (1).
(2) Assume that $M = G/K$ carries a G-invariant metric connection ∇ which is traceless cyclic. This means that the invariant tensor $A(X, Y, Z)$ must satisfy the conditions

$$\mathfrak{S}_{X,Y,Z}A(X, Y, Z) = 0 \ \text{ and } \ \sum_i A(Z_i, Z_i, Z) = 0, \tag{$\dagger$}$$

where $\{Z_i\}$ is an orthonormal basis of $\mathfrak{m}$ with respect to $\langle\ ,\ \rangle$. By (3.3) we see that

$$\sum_i A(Z_i, Z_i, Z) = 0 \ \Leftrightarrow\ \sum_i \hat{\mu}(Z_i, Z_i, Z) = 0.$$

However, $\sum_i \hat{\mu}(Z_i, Z_i, Z) = \sum_i \langle \mu(Z_i, Z_i), Z\rangle = \langle \sum_i \mu(Z_i, Z_i), Z\rangle = \langle \sum_i \Lambda(Z_i)Z_i, Z\rangle$, where $\Lambda \equiv \Lambda^{\mu} : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ is the associated connection map. Thus, the traceless condition in (†) holds if and only if $\sum_i \mu(Z_i, Z_i) = \sum_i \Lambda(Z_i)Z_i = 0$. Now, for the cyclic condition in (†), using (3.3) we obtain the relation

$$\mathfrak{S}_{X,Y,Z}A(X, Y, Z) = \mathfrak{S}_{X,Y,Z}\hat{\mu}(X, Y, Z) - \frac{3}{2}\langle [X, Y]_{\mathfrak{m}}, Z\rangle$$

and in this way we conclude the second stated relation. In fact, this follows also by the cyclic sum $\mathfrak{S}_{X,Y,Z}T(X, Y, Z) = 0$, where T is the torsion of ∇.
(3) Parts (3) and (4) are immediate due to the description given in (2) and the definition of the classes $\mathcal{A}_1 \oplus \mathcal{A}_2$, and $\mathcal{A}_2 \oplus \mathcal{A}_3$. □

Remark 3.19. If $(M = G/K, g)$ is an effective Riemannian symmetric space endowed with a G-invariant metric connection $\nabla \equiv \nabla^{\mu}$ corresponding to some $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, then the conclusions in Proposition 3.17 are simplified, i.e. for the tensor $A = \nabla^{\mu} - \nabla^{g}$ we deduce that

- $A \in \mathcal{A}_1$, i.e. ∇ is vectorial, if and only if $\exists$ a global G-invariant 1-form φ on M such that
$$\hat{\mu}(X, Y, Z) = \langle X, Y\rangle\varphi(Z) - \langle X, Z\rangle\varphi(Y), \quad \forall\, X, Y, Z \in \mathfrak{m}.$$
- $A \in \mathcal{A}_2$, i.e. ∇ is traceless cyclic, if and only if $\mathfrak{S}_{X,Y,Z}\hat{\mu}(X, Y, Z) = 0$ and $\sum_i \Lambda(Z_i)Z_i = 0$.
- $A \in \mathcal{A}_1 \oplus \mathcal{A}_2$, i.e. ∇ is cyclic, if and only if $\mathfrak{S}_{X,Y,Z}\hat{\mu}(X, Y, Z) = 0$ for any $X, Y, Z \in \mathfrak{m}$.

Because on a compact Riemannian symmetric space $(M = G/K, g)$ of Type I, the G-invariant metric connections are exhausted by the torsion-free canonical connection $\nabla^{c} = \nabla^{g}$ associated to $\mathfrak{m}$, in the compact case the above conditions are of particular interest for compact connected (non-simple) Lie groups endowed with a bi-invariant metric, where $A = \nabla^{\mu} - \nabla^{g}$ can be non-trivial. For example, below we apply these considerations for the Lie group U_n. Finally notice that considering a naturally reductive space as in Proposition 3.17 (or even a symmetric space as above), it is easy to certify that any G-invariant metric connection of type $\mathcal{A}_3$ is also of type $\mathcal{A}_2 \oplus \mathcal{A}_3$, any G-invariant metric connection of type $\mathcal{A}_1$ it is also of type $\mathcal{A}_1 \oplus \mathcal{A}_2$, etc.

Proposition 3.20. *For $n \geq 3$, the bi-invariant metric connection ∇^{μ} on $(\mathrm{U}_n, \langle\ ,\ \rangle)$ induced by the map $\mu := \mu_4 - \mu_5$ of Theorem 3.15 has torsion of vectorial type.*

Proof. The Lie group U_n has 1-dimensional centre Z; hence the quotient $(\mathrm{U}_n \times \mathrm{U}_n)/\Delta\mathrm{U}_n$ is not yet effective, but the expression $(\mathrm{U}_n/Z)/(\Delta\mathrm{U}_n/\Delta Z)$ gives rise to an effective coset space. From now on we shall identify $\mathrm{U}_n \cong (\mathrm{U}_n \times \mathrm{U}_n)/\Delta\mathrm{U}_n \cong (\mathrm{U}_n/Z)/(\Delta\mathrm{U}_n/\Delta Z)$ and write $\mathfrak{u}(n) \oplus \mathfrak{u}(n) = \Delta\mathfrak{u}(n) \oplus \mathfrak{m}$ for the associated symmetric reductive decomposition, where

$$\Delta\mathfrak{u}(n) := \{(X, X) \in \mathfrak{u}(n) \oplus \mathfrak{u}(n) : X \in \mathfrak{u}(n)\}, \quad \mathfrak{m} := \{(X, -X) \in \mathfrak{u}(n) \oplus \mathfrak{u}(n) : X \in \mathfrak{u}(n)\}$$

are both isomorphic to $\mathfrak{u}(n)$ as U_n-modules. Because any compact connected Lie group G endowed with a bi-invariant metric is a compact normal homogeneous space and moreover a compact symmetric space, the condition $\mathfrak{g} = \tilde{\mathfrak{g}}$ of Proposition 3.17 is satisfied and we can apply the considerations of Remark 3.19. Consider the Lie algebra $\mathfrak{u}(n)$ endowed with the bilinear mapping $\mu(X, Y) = i(\mathrm{tr}(Y)X - \mathrm{tr}(XY)\,\mathrm{Id})$, given in Theorem 3.15. Since $\langle X, Y\rangle = -\mathrm{tr}(XY)$ we conclude that

$$\begin{aligned}\hat{\mu}(X, Y, Z) := \langle \mu(X, Y), Z\rangle &= i\,\mathrm{tr}(Y)\langle X, Z\rangle - i\,\mathrm{tr}(XY)\langle \mathrm{Id}, Z\rangle \\ &= i\,\mathrm{tr}(Y)\langle X, Z\rangle + i\langle X, Y\rangle\langle \mathrm{Id}, Z\rangle \\ &= i\,\mathrm{tr}(Y)\langle X, Z\rangle - i\,\mathrm{tr}(Z)\langle X, Y\rangle,\end{aligned} \tag{3.4}$$

for any $X, Y, Z \in \mathfrak{u}(n)$. Consider now the 1-form $\varphi : \mathfrak{u}(n) \to \mathbb{R}$, $Y \mapsto \varphi(Y) := -i\,\mathrm{tr}(Y)$. It is easy to see that φ is a U_n-invariant 1-form with kernel $\mathfrak{su}(n)$. But then, based on (3.4) we obtain that

$$\hat{\mu}(X, Y, Z) = -\langle X, Z\rangle\varphi(Y) + \langle X, Y\rangle\varphi(Z),$$

for any $X, Y, Z \in \mathfrak{u}(n)$ and using Remark 3.19 we conclude that $\mathcal{A}^{\mu} := \nabla^{\mu} - \nabla^{g} \in \mathcal{A}_1$. □

Remark 3.21. By Theorem 3.15, the group U_n ($n \geq 3$) is equipped with a two dimensional space of bi-invariant metric connections ∇^{f}, given by the bilinear map $f := a\mu_1 + b\mu$ ($a, b \in \mathbb{R}$) where μ_1 and μ are given by (3.2) and Theorem 3.15, respectively. In general, ∇^{f} is of mixed type $\mathcal{A}_1 \oplus \mathcal{A}_3$, but the conditions that the type of ∇^{f} is either purely $\mathcal{A}_3$ or purely $\mathcal{A}_1$, naturally defines the one dimensional subfamilies $a\mu_1$ and $b\mu$, respectively. Thus, we can express the space of bilinear mappings inducing bi-invariant metric connections on U_n as a direct sum of these families.

3.5. The curvature tensor and the Ricci tensor

Let us now examine the curvature tensor.

Proposition 3.22. *Let $(M = G/K, g)$ be a naturally reductive Riemannian manifold as in Proposition 3.17. Then, the curvature tensor $R^{\nabla^{\mu}} \equiv R^{\nabla}$ associated to a G-invariant metric connection $\nabla \equiv \nabla^{\mu}$ on $M = G/K$, induced by some $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, satisfies the following relation*

$$\begin{aligned} R^{\nabla}(X, Y)Z &= R^{g}(X, Y)Z + A(X, \mu(Y, Z)) - A(Y, \mu(X, Z)) - A([X, Y]_{\mathfrak{m}}, Z) \\ &\quad + \frac{1}{2}\Big([X, A(Y, Z)]_{\mathfrak{m}} - [Y, A(X, Z)]_{\mathfrak{m}}\Big), \end{aligned}$$

for any $X, Y, Z \in \mathfrak{m}$, where the tensor A is defined by the difference $A = \nabla - \nabla^{g}$ and R^{g} is the Riemannian curvature tensor. If $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair, then the last three terms in the previous relation are cancelled.

Proof. The proof relies on a straightforward computation using the formulas

$$R^{\nabla}(X, Y)Z = \mu(X, \mu(Y, Z)) - \mu(Y, \mu(X, Z)) - \mu([X, Y]_{\mathfrak{m}})Z - [[X, Y]_{\mathfrak{k}}, Z],$$

and $A(X, Y) = \mu(X, Y) - \mu^{g}(X, Y) = \Lambda(X)Y - \Lambda^{g}(X)Y$ where $\mu^{g}(X, Y) = \Lambda^{g}(X)Y = \frac{1}{2}[X, Y]_{\mathfrak{m}}$ is the bilinear map associated to the Levi-Civita connection on $M = G/K$, see also (3.3). The last conclusion relies on the symmetric reductive decomposition, in particular (3.3) reduces to $A(X, Y, Z) = \hat{\mu}(X, Y, Z)$ for any $X, Y, Z \in \mathfrak{m}$. $\square$

Consider now a G-invariant metric connection ∇ of vectorial type. Let us denote by φ the associated $\mathrm{Ad}(K)$-invariant 1-form on $\mathfrak{m}$ and by $\xi \in \mathfrak{m}$ the dual vector with respect to $\langle\ ,\ \rangle$. If $\|\xi\|^2 \neq 0$, then ∇ is called a *G-invariant connection of non-degenerate vectorial type*. In this case, by applying [7, Corol. 3.1] or by a direct calculation based on Proposition 3.22, we get that

Corollary 3.23. *Let $(M = G/K, g)$ be an n-dimensional naturally reductive manifold as in Proposition 3.17, endowed with a G-invariant metric connection $\nabla \equiv \nabla^{\mu}$ of non-degenerate vectorial type. Then*
(1) For any $X, Y \in \mathfrak{m}$, the Ricci tensor Ric^{∇} associated to ∇ satisfies the relation

$$\mathrm{Ric}^{\nabla}(X, Y) = \mathrm{Ric}^{g}(X, Y) + (n-2)\langle X, \xi\rangle\langle Y, \xi\rangle + (2-n)\|\xi\|^2\langle X, Y\rangle + \frac{2-n}{2}\langle [X, Y]_{\mathfrak{m}}, \xi\rangle. \tag{3.5}$$

(2) Ric^{∇} is symmetric if and only if $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair and this is equivalent to say that φ is a closed invariant 1-form.

Proof. We prove only the second claim. By (3.5) it follows that

$$\mathrm{Ric}^{\nabla}(X, Y) - \mathrm{Ric}^{\nabla}(Y, X) = (n-2)\langle [X, Y]_{\mathfrak{m}}, \xi\rangle, \quad \forall\, X, Y \in \mathfrak{m}.$$

Hence, Ric^{∇} is symmetric if and only if $\langle [X, Y]_{\mathfrak{m}}, \xi\rangle = 0$. But since $\xi \neq 0$, this is equivalent to say that $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair, i.e. $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. By the definition of the differential of an invariant form (cf. [47, pp. 248–250]), or by [7, Prop. 3.2] we get the last correspondence. $\square$

Specializing to the Lie group U_n we conclude that

Corollary 3.24. *Consider the Lie group U_n $(n \geq 3)$ endowed with the bi-invariant metric connection ∇^{μ} induced by the map $\mu = \mu_4 - \mu_5$, as described in Theorem 3.15. Then, the Ricci tensor Ric^{μ} associated to ∇^{μ} is given by the following symmetric invariant bilinear form on $\mathfrak{u}(n)$:*

$$\mathrm{Ric}^{\mu}(X, Y) = \frac{1}{2}\Big\{(2n^2 - n - 4)\,\mathrm{tr}\,XY + (5 - 2n^2)\,\mathrm{tr}\,X\,\mathrm{tr}\,Y\Big\},$$

for any $X, Y \in \mathfrak{m} \cong \mathfrak{u}(n)$.

Proof. We use the notation of Proposition 3.20 and view U_n as an effective symmetric space endowed with the bi-invariant metric induced by $\langle X, Y\rangle = -\mathrm{tr}(XY)$. Consider the Nomizu map

$$\Lambda^{\mu}(X)Y := i(\mathrm{tr}\,(Y)X - \mathrm{tr}\,(XY)\,\mathrm{Id}), \quad \forall\, X, Y \in \mathfrak{m} \cong \mathfrak{u}(n).$$

By Proposition 3.20 we know that the bi-invariant metric connection $\nabla^{\mu}_X Y = \nabla^{c}_X Y + \Lambda^{\mu}(X)Y$ has torsion of vectorial type, associated to the U_n-invariant linear form $\varphi(Z) = -i\,\mathrm{tr}(Z) = i\langle \mathrm{Id}, Z\rangle$. The dual vector $\xi \in \mathfrak{m}$ is defined by $\varphi(Z) = \langle Z, \xi\rangle$ for any $Z \in \mathfrak{m}$ and hence we conclude that $\xi = i\,\mathrm{Id}$, in particular $0 \neq \langle \xi, \xi\rangle = 1 = \|\xi\|^2$. Thus, the vectorial structure is non-degenerate and we can apply Corollary 3.23, i.e.

$$\begin{aligned} \mathrm{Ric}^{\mu}(X, Y) &= \mathrm{Ric}^{g}(X, Y) + (n^2 - 2)\big(\langle X, \xi\rangle\langle Y, \xi\rangle - \langle X, Y\rangle\big) \\ &= \mathrm{Ric}^{g}(X, Y) + (n^2 - 2)\big(\mathrm{tr}(XY) - \mathrm{tr}\,X\,\mathrm{tr}\,Y\big). \end{aligned}$$

Now, $\langle\ ,\ \rangle$ is a bi-invariant inner product and hence $\mathrm{Ric}^{g}(X, Y) = -\frac{1}{4}B(X, Y)$ for any $X, Y \in \mathfrak{u}(n)$, where $B(X, Y) = 2n\,\mathrm{tr}\,XY - 2\,\mathrm{tr}\,X\,\mathrm{tr}\,Y$ is the Killing form of U_n (cf. [13,10] where the statement is given for a compact semi-simple Lie group, but notice that Ric^{g} satisfies the same formula for any bi-invariant metric g on a Lie group G). Thus, a small computation in combination with the formula given above yields the result. $\square$

4. Classification of invariant connections on non-symmetric SII spaces

4.1. Strongly isotropy irreducible spaces (SII)

Consider a compact, connected, effective, non-symmetric SII homogeneous space $M = G/K$. Since G is a compact simple Lie group (see [46, p. 62]), any such manifold is a standard homogeneous Riemannian manifold. Passing to a covering $\tilde{G}$ of G, if G/K is not simply-connected but G is connected, then $\tilde{G}$ acts transitively on the universal covering of G/K with connected isotropy group, say K', and it turns out that G/K is SII if and only if $\tilde{G}/K'$ is. Hence, whenever necessary we can assume that G/K is a compact, connected and simply-connected, effective, non-symmetric SII space, with G being compact, connected and simple and $K \subset G$ compact and connected. In this setting, the strongly isotropy irreducible condition is equivalent to an (almost effective) irreducible action of the Lie algebra $\mathfrak{k} = T_eK$ on $\mathfrak{m} \cong T_oG/K$. For a list of non-symmetric SII spaces we refer to [13, Tables 5, 6, p. 203]. We remark however that there are misprints in Table 6 of [13], related to SII homogeneous spaces $M = G/K$ of $G = \mathrm{Sp}_n$ (compare for example with [46, Thm. 7.1]). We correct these errors in our Table 5.

Proposition 4.1. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective, non-symmetric (compact) SII homogeneous Riemannian manifold, endowed with a G-invariant affine connection ∇^{μ} compatible with the Killing metric $\langle\ ,\ \rangle = -B|_{\mathfrak{m}}$, where $\mu \in \mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m}, \mathfrak{m})$. Then, the torsion T^{μ} of ∇^{μ} does not carry a component of vectorial type.*

Proof. Assume that $M = G/K$ carries a G-invariant metric connection ∇ whose torsion is of vectorial type and let $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$ be the reductive decomposition with respect to the Killing metric. Then, by Proposition 3.17(1), we have that $\hat{\mu}(X,Y,Z) = \frac{1}{2}\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle X,Y\rangle\varphi(Z) - \langle X,Z\rangle\varphi(Y)$, for some G-invariant 1-form φ on $M = G/K$. However, $\mathfrak{m}$ is a self-dual and (strongly) irreducible K-module over $\mathbb{R}$; thus global G-invariant 1-forms do not exist, since dually the isotropy representation needs to preserve some vector field ξ and hence a 1-dimensional subspace of $\mathfrak{m}$, spanned by ξ. □

Corollary 4.2. *Let $(M = G/K, g)$ be an effective, non-symmetric (compact) SII homogeneous Riemannian manifold, endowed with a non torsion-free G-invariant metric connection ∇. Then, the torsion $0 \neq T$ of ∇ is totally skew-symmetric, $T \in \mathcal{A}_3 \cong \Lambda^3TM$, or traceless cyclic $T \in \mathcal{A}_2$, or of mixed type $T \in \mathcal{A}_2\oplus\mathcal{A}_3$, i.e. traceless.*

4.2. An application in the spin case

Consider an effective, non-symmetric (compact) SII homogeneous Riemannian manifold $(M^n = G/K, g)$. Assume that $M = G/K$ admits a G-invariant spin structure, i.e. a G-homogeneous $\mathrm{Spin}(\mathfrak{m})$-principal bundle $P \to M$ and a double covering morphism $\Lambda : P \to \mathrm{SO}(M,g)$ compatible with the principal groups' actions. Recall that an invariant spin structure corresponds to a lift of the isotropy representation χ into the spin group $\mathrm{Spin}(\mathfrak{m}) \equiv \mathrm{Spin}_n$, i.e. a homomorphism $\widetilde{\chi} : K \to \mathrm{Spin}(\mathfrak{m})$ such that $\chi = \lambda\circ\widetilde{\chi}$, where $\lambda : \mathrm{Spin}(\mathfrak{m}) \to \mathrm{SO}(\mathfrak{m})$ is the double covering of $\mathrm{SO}(\mathfrak{m}) \equiv \mathrm{SO}_n$. We shall denote by $\kappa_n : \mathrm{C}\ell^{\mathbb{C}}(\mathfrak{m}) \xrightarrow{\sim} \mathrm{End}(\Delta_{\mathfrak{m}})$ the Clifford representation and by $\mathrm{C}\ell(X\otimes\phi) := \kappa_n(X)\psi = X\cdot\psi$ the Clifford multiplication between vectors and spinors, see [1] for more details. Set $\rho := \kappa\circ\widetilde{\chi} : K \to \mathrm{Aut}(\Delta_{\mathfrak{m}})$, where $\kappa = \kappa_n|_{\mathrm{Spin}(\mathfrak{m})}$: $\mathrm{Spin}(\mathfrak{m}) \to \mathrm{Aut}(\Delta_{\mathfrak{m}})$ is the spin representation. The spinor bundle $\Sigma \to G/K$ is the homogeneous vector bundle associated to $P := G\times_{\widetilde{\chi}}\mathrm{Spin}(\mathfrak{m})$ via the representation ρ, i.e. $\Sigma = G\times_{\rho}\Delta_{\mathfrak{m}}$. Therefore we may identify sections of Σ with smooth functions $\varphi : G \to \Delta_{\mathfrak{m}}$ such that $\varphi(gk) = \kappa\big(\widetilde{\chi}(k^{-1})\big)\varphi(g) = \rho(k^{-1})\varphi(g)$ for any $g \in G$, $k \in K$.

Choose a G-invariant metric connection ∇ on G/K, corresponding to a connection map $\Lambda \in \mathrm{Hom}_K(\mathfrak{m}, \mathfrak{so}(\mathfrak{m}))$. The lift $\widetilde{\Lambda} := \lambda_*^{-1}\circ\Lambda : \mathfrak{m} \to \mathfrak{spin}(\mathfrak{m})$ induces a covariant derivative on spinor fields (which we still denote by the same symbol) $\nabla : \Gamma(\Sigma) \to \Gamma(T^*(G/K)\otimes\Sigma)$, given by $\nabla_X\psi = X(\psi) + \widetilde{\Lambda}(X)\psi$. Here, the vector $X \in \mathfrak{m}$ is considered as a left-invariant vector field in G and $\widetilde{\Lambda}(X)\psi$ as an equivariant function $\widetilde{\Lambda}(X)\psi : G \to \mathfrak{m}$. The Dirac operator $D := \mathrm{C}\ell\circ\nabla : \Gamma(\Sigma) \to \Gamma(\Sigma)$ associated to ∇ is defined as follows (cf. [1]):

$$D(\psi) := \sum_i \kappa_n(Z_i)\{Z_i(\psi) + \widetilde{\Lambda}(Z_i)\psi\} = \sum_i Z_i\cdot\{Z_i(\psi) + \widetilde{\Lambda}(Z_i)\psi\},$$

where Z_i denotes a $\langle\ ,\ \rangle$-orthonormal basis of $\mathfrak{m}$.

Remark 4.3. Given a spin Riemannian manifold (M, g) endowed with a metric connection ∇, basic properties of the induced Dirac operator $D = \mathrm{C}\ell\circ\nabla$ are reflected in the type of the torsion of ∇. For example, by a result of Th. Friedrich [24] (see also [26,40]), it is known that the formal self-adjointness of the Dirac operator $D = \mathrm{C}\ell\circ\nabla$ is equivalent to the condition $A \in \mathcal{A}_2\oplus\mathcal{A}_3$, where $A = \nabla - \nabla^g$. Hence, in our case as an immediate consequence of Corollary 4.2 we obtain that

Corollary 4.4. *Let $(M = G/K, g)$ be an effective, non-symmetric (compact) SII homogeneous Riemannian manifold, endowed with a G-invariant metric connection ∇ and a G-invariant spin structure. Then, the Dirac operator D associated to ∇ is formally self-adjoint.*

Note that the classification of invariant spin structures on non-symmetric SII spaces is an open problem (see [15] for invariant spin structures on symmetric spaces and [8] for a more recent study of spin structures on reductive homogeneous spaces).

4.3. Classification results on invariant connections

For the presentation of the classification results, we use the notation of [39, p. 299]. In particular, for a compact simple Lie group G we shall denote by $R(\pi)$ the complex irreducible representation of highest weight π. We mention that the isotropy representation of a compact, non-symmetric, effective SII space turns out to be of either real or complex type. In fact, fixing a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, whenever the complexification $\mathfrak{m}^{\mathbb{C}}$ splits into two complex submodules, these are never equivalent representations (see also [46]). Hence, by Schur's lemma we have the identification $\mathrm{Hom}_K(\mathfrak{m}, \mathfrak{m}) = \mathbb{C}$ for complex type and $\mathrm{Hom}_K(\mathfrak{m}, \mathfrak{m}) = \mathbb{R}$ for real type. In the first case, the endomorphism J induced by $i \in \mathbb{C}$ makes G/K a homogeneous almost complex manifold. Note that the same conclusions are true for a symmetric space, see [32,33] (recall that the adjoint representation of a compact simple Lie group is always of real type).

Remark 4.5. The multiplicities that we describe below have also been presented in the PhD thesis [19] (see Tables I.3.1–I.3.4, pp. 77–79), for a different however aim, namely the description of the components of the intrinsic torsion associated to (irreducible) G-structures over non-symmetric compact SII spaces (see also [20]). We remark that there are a few errors/omissions in [19], related with some low-dimensional cases, namely:

- the case $p = 2$, $q \geq 3$ of the family $\mathrm{SU}_{pq}/\mathrm{SU}_p \times \mathrm{SU}_q$,
- the case $n = 5$ of the family $\mathrm{SU}_{\frac{n(n-1)}{2}}/\mathrm{SU}_n$,
- the case $n = 6$ of the family $\mathrm{SO}_{\frac{(n-1)(n+2)}{2}}/\mathrm{SO}_n$ (due to isomorphism $\mathfrak{so}(6) = \mathfrak{su}(4)$).

In these mentioned cases, the general decompositions of $\Lambda^2\mathfrak{m}$ or $\mathrm{Sym}^2\,\mathfrak{m}$ change and most times this affects to multiplicities that we are interested in. Notice also that for the manifold $\mathrm{SO}_{4n}/\mathrm{Sp}_n \times \mathrm{Sp}_1$ the enumeration in [46,13] starts for $n \geq 2$ (as we do), but in [19] it is written $n \geq 3$. We correct these errors in our Table 4 (they are indicated by an asterisk). Notice finally that the author of this thesis uses the LiE program (as we do) and for infinite families he is based on stability arguments, see [19, Rem. I.3.9]. Below we also give examples of how such families can be treated even without the aid of a computer.

Remark 4.6. Given a reductive homogeneous space $M = G/K$ of a classical simple Lie group G, there is a simple method for the computation of the associated isotropy representation $\chi : K \to \mathrm{Aut}(\mathfrak{m})$, given as follows. Let us denote by $\rho_n : \mathrm{SO}_n \to \mathrm{Aut}(\mathbb{R}^n)$, $\mu_n : \mathrm{SU}_n \to \mathrm{Aut}(\mathbb{C}^n)$ and $\nu_n : \mathrm{Sp}_n \to \mathrm{Aut}(\mathbb{H}^n)$ be the standard representations of SO_n, SU_n (or U_n), and Sp_n, respectively. Recall that the complexified adjoint representation $\mathrm{Ad}^{\mathbb{C}}_G = \mathrm{Ad}_G \otimes \mathbb{C}$, satisfies

$$\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SO}_n} = \Lambda^2\rho_n, \quad \mathrm{Ad}^{\mathbb{C}}_{\mathrm{U}_n} = \mu_n \otimes \mu_n^*, \quad \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_n} \oplus 1 = \mu_n \otimes \mu_n^*, \quad \mathrm{Ad}^{\mathbb{C}}_{\mathrm{Sp}_n} = \mathrm{Sym}^2\nu_n,$$

where μ_n^* is the dual representation of μ_n and 1 denotes the trivial 1-dimensional representation. Let G be one of the Lie groups SO_n, SU_n, Sp_n and let $\pi : K \to G$ be an (almost) faithful representation of a compact connected subgroup K. Using the identity $\mathrm{Ad}\big|_K = \mathrm{Ad}_K \oplus \chi$, we see that the isotropy representation χ of $G/\pi(K)$ is determined by $\Lambda^2\pi = \mathrm{ad}_{\mathfrak{k}} \oplus \chi$ in the orthogonal case, by $\pi \otimes \pi^* = 1 \oplus \mathrm{ad}_{\mathfrak{k}} \oplus \chi$ in the unitary case and finally by $\mathrm{Sym}^2\,\pi = \mathrm{ad}_{\mathfrak{k}} \oplus \chi$ in the symplectic case (see [10,46]).

Theorem 4.7. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective, non-symmetric (compact) SII homogeneous space. Consider the B-orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then, the complexified isotropy representation $\mathfrak{m}^{\mathbb{C}}$ and the multiplicities $\mathbf{a}$, $\mathbf{s}$, $\mathcal{N}$ and ℓ are given in Tables 4 and 5.*

4.4. On the Theorems A.1, A.2 and B—Conclusions

The results in Tables 4 and 5 allow us to deduce that several non-symmetric SII spaces are carrying *new* families of invariant metric connections, in the sense that they are different from the Lie bracket family $\eta^{\alpha}(X, Y) = \frac{1-\alpha}{2}[X, Y]_{\mathfrak{m}}$ (see Lemma 3.4). In combination with Lemma 3.12, we also certify the existence of compact, effective, non-symmetric SII quotients $M = G/K$ which are endowed with additional families of G-invariant metric connections with skew-torsion, besides η^{α}. In full details, this occurs in the following two situations:

- when $2 \leq \ell \leq \mathbf{a}$ and the isotropy representation is not of complex type (since for complex type we may have $\ell = 2 = \mathbf{a}$, but due to Schur's lemma all these invariant connections must be exhausted by the family $\eta^{\alpha}(X, Y) = \frac{1-\alpha}{2}[X, Y]_{\mathfrak{m}}$ with $\alpha \in \mathbb{C}$), or
- when the isotropy representation is of complex type but ℓ (and hence $\mathbf{a}$) is strictly greater than 2.

This observation, in combination with Lemmas 3.7, 3.12 and the results in Tables 4 and 5, yields Theorems A.1 and A.2. Theorem B it is also a direct conclusion of the multiplicity $\mathbf{s}$ given in Tables 4 and 5 and Lemma 3.9. In fact, for affine connections induced by symmetric elements $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$, we also conclude that

Corollary 4.8. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective, non-symmetric, SII homogeneous space associated to the Lie group $G = \mathrm{SU}_n$. Then, there is always a copy of $\mathfrak{m}$ inside $\mathrm{Sym}^2\,\mathfrak{m}$, induced by the restriction of the $\mathrm{Ad}(\mathrm{SU}_n)$-invariant symmetric bilinear* *mapping*

$$\eta : \mathfrak{su}(n) \times \mathfrak{su}(n) \to \mathfrak{su}(n), \quad \eta(X, Y) := i\big\{XY + YX - \frac{2}{n}\operatorname{tr}(XY)\operatorname{Id}\big\}$$

on the corresponding reductive complement $\mathfrak{m}$. *If* M *is isometric to one of the manifolds*

$$\mathrm{SU}_{10}/\mathrm{SU}_5, \quad \mathrm{SU}_{2q}/\mathrm{SU}_2 \times \mathrm{SU}_q\ (q \geq 3), \quad \mathrm{SU}_{16}/\mathrm{Spin}_{10},$$

then the 1-parameter family of SU_n*-invariant affine connections on* $M = \mathrm{SU}_n/K$ *associated to the restriction* $\eta|_{\mathfrak{m}}: \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$, *exhausts all* SU_n*-invariant affine connections induced by some* $0 \neq \mu \in \operatorname{Hom}_K(\operatorname{Sym}^2 \mathfrak{m}, \mathfrak{m})$.

Proof. The first part is based on [33, Thm. 6.1]. Notice that η is known by [32, p. 550]. Now, using the results of Tables 4 and 5 about the multiplicity $\mathbf{s}$ of $\mathfrak{m}$ inside $\operatorname{Sym}^2 \mathfrak{m}$, we obtain the result. $\square$

4.5. Some explicit examples

Let us now compute the desired multiplicities $\mathbf{a}$, $\mathbf{s}$ and ℓ, for general families of (non-symmetric) SII spaces, without the aid of computer. For this we need first to recall some preliminaries of representation theory (for more details we refer to [14,42,19]).

If π is a complex representation of a compact Lie algebra $\mathfrak{k}$, then $\overline{\pi} \cong \pi^*$, where $\overline{\pi}$ denotes the complex conjugate representation and π^* the dual representation. If π is a complex representation of $\mathfrak{k}$ on V, then there is a symmetric (resp. skew-symmetric) non-degenerate bilinear form on V invariant under π, if and only if there is an anti-linear intertwining map τ with $\tau^2 = \operatorname{Id}$ (resp. $\tau^2 = -\operatorname{Id}$) [14, Prop. 6.4.]. If π is irreducible then Schur's lemma ensures the uniqueness of such a bilinear form. A complex representation carrying a conjugate linear intertwining map τ with $\tau^2 = \operatorname{Id}$ (resp. $\tau^2 = -\operatorname{Id}$) is called of real type (resp. quaternionic type). Finally we call a complex representation $\pi : \mathfrak{k} \to \mathfrak{gl}(V)$ of complex type if it is not self dual, i.e. $V \ncong V^*$.

Let (π, V) and (π', W) be representations of a connected (not necessarily compact) Lie group H on two vector spaces V and W, respectively. It is important to note that even if π and π' are irreducible, then the tensor product representation $V \otimes W$, defined by $\pi \otimes \pi' : H \to \operatorname{Aut}(V \otimes W)$, $(\pi \otimes \pi')(h)(u \otimes w) = \pi(h)u \otimes \pi'(h)w$, is always reducible. Let us denote by $\Lambda^k\pi$ and $\operatorname{Sym}^k \pi$ the kth exterior power and kth symmetric power, respectively. For $k = 2$, it is easy to prove that

$$\begin{cases} \Lambda^2(V \oplus W) & = & \Lambda^2 V \oplus (V \otimes W) \oplus \Lambda^2 W, \\ \operatorname{Sym}^2(V \oplus W) & = & \operatorname{Sym}^2 V \oplus (V \otimes W) \oplus \operatorname{Sym}^2 W. \end{cases} \tag{4.1}$$

If (π, V) and (π', W) are representations of two connected Lie groups H and H', respectively, then the vector space $V \otimes W$ carries a representation of the product group $H \times H'$, say $(\pi \hat{\otimes} \pi', V \otimes W)$, given by $\pi \hat{\otimes} \pi'(h, h')(u \otimes w) = \pi(h)u \otimes \pi'(h')w$. This representation is called the external tensor product of π and π'. In the finite-dimensional case, $\pi \hat{\otimes} \pi'$ is an irreducible representation of $H \times H'$, if and only π and π' are both irreducible. In particular, if H, H' are compact Lie groups, then a representation of $H \times H'$ in $\mathrm{GL}(\mathbb{C}^n)$ is irreducible if and only if it is the tensor product of an irreducible representation of H with one of H'. Finally, one has the following equivariant isomorphisms:

$$\begin{cases} (V \hat{\otimes} W) \otimes (V' \hat{\otimes} W') & = & (V \otimes V') \hat{\otimes} (W \otimes W') \\ \Lambda^2(V \hat{\otimes} W) & = & (\Lambda^2 V \hat{\otimes} \operatorname{Sym}^2 W) \oplus (\operatorname{Sym}^2 V \hat{\otimes} \Lambda^2 W) \\ \operatorname{Sym}^2(V \hat{\otimes} W) & = & (\operatorname{Sym}^2 V \hat{\otimes} \operatorname{Sym}^2 W) \oplus (\Lambda^2 V \hat{\otimes} \Lambda^2 W) \end{cases} \tag{4.2}$$

We finally remark that if V, W are complex irreducible representations of two compact Lie groups H and H' respectively, then $V \hat{\otimes} W$ is of real type if V, W are both of real type or both of quaternionic type, $V \hat{\otimes} W$ is if complex type if at least one of V, W are of complex type and finally, $V \hat{\otimes} W$ is of quaternionic type if one of V, W is of real type and the other one of quaternionic type.

Lemma 4.9. *Consider the homogeneous space* $M_{p,q} := G/K = \mathrm{SU}_{pq}/\mathrm{SU}_p \times \mathrm{SU}_q$ *with* $p, q > 2$, $p+q > 4$. *Then, the multiplicities of the isotropy representation* $\mathfrak{m} = \mathrm{Ad}_{\mathrm{SU}_p} \hat{\otimes} \mathrm{Ad}_{\mathrm{SU}_q}$ *inside* $\Lambda^2\mathfrak{m}$ *and* $\operatorname{Sym}^2 \mathfrak{m}$ *are given as follows: for* $p = 2$, $q \geq 3$ *it is* $\mathbf{a} = \mathbf{s} = 1$, *while for* $p, q \geq 3$ *it is* $\mathbf{a} = \mathbf{s} = 2$. *Moreover, the dimension of the trivial submodule* $(\Lambda^3\mathfrak{m})^K$ *is* $\ell = 1$ *for* $p = 2$, $q \geq 3$ *and* $\ell = 2$ *for* $p, q \geq 3$.

Proof. The inclusion $\pi : K \to G$ is given by the external tensor product of the standard representations μ_p and μ_q of SU_p and SU_q, respectively. Thus, a short application of Remark 4.6 in combination with the relation $\mathrm{Ad}^{\mathbb{C}}_{A_n} = R(\pi_1 + \pi_n)$, yields that

$$\mathfrak{m}^{\mathbb{C}} = (\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p} \hat{\otimes} \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}) = R(\pi_1 + \pi_{p-1}) \hat{\otimes} R(\pi_1 + \pi_{q-1}).$$

Consequently, the isotropy representation $\mathfrak{m} = \mathrm{Ad}_{\mathrm{SU}_p} \hat{\otimes} \mathrm{Ad}_{\mathrm{SU}_q}$ of $M_{p,q} = \mathrm{SU}_{pq}/\mathrm{SU}_p \times \mathrm{SU}_q$ is irreducible over $\mathbb{R}$ and is of real type, since it is the external tensor product of two representations of real type. Now, by [39] we also know that

$$\Lambda^2 \mathrm{Ad}^{\mathbb{C}}_{A_n} = R(2\pi_1 + \pi_{n-1}) \oplus R(\pi_2 + 2\pi_n) \oplus \mathrm{Ad}^{\mathbb{C}}_{A_n}, \quad n \geq 3 \tag{4.3}$$

$$\mathrm{Sym}^2\,\mathrm{Ad}^{\mathbb{C}}_{A_n} = \begin{cases} R(2\pi_1+2\pi_n)\oplus R(\pi_2+\pi_{n-1})\oplus \mathrm{Ad}^{\mathbb{C}}_{A_n}\oplus 1, & \text{if } n\geq 3,\\ R(2\pi_1+2\pi_2)\oplus \mathrm{Ad}^{\mathbb{C}}_{A_n}\oplus 1, & \text{if } n=2.\end{cases} \tag{4.4}$$

Certainly, for $\mathfrak{su}_2=\mathfrak{so}_3=\mathfrak{sp}_1$ one gets a 3-dimensional irreducible representation $\Lambda^2(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2})\cong \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}=R(2\pi_1)$. Moreover, it is $\mathrm{Sym}^2(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2})=R(4\pi_1)\oplus 1$. Notice also that $\Lambda^2(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3})=R(3\pi_1)\oplus R(3\pi_2)\oplus \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}$, since $\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}=R(\pi_1+\pi_2)$. Due to this small speciality of SU_2 and the different decomposition of $\mathrm{Sym}^2\,\mathrm{Ad}^{\mathbb{C}}_{A_n}$ (for $n=2$ and $n\geq 3$, respectively) one has to separate the examination into two cases:

Case A: $p=2$, $q\geq 3$. Then we have $M_{2q}=\mathrm{SU}_{2q}/\mathrm{SU}_2\times\mathrm{SU}_q$ and

$$\mathfrak{m}^{\mathbb{C}}=\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}=R(2\pi_1)\hat{\otimes}R(\pi_1+\pi_{q-1}),\quad (q\geq 3).$$

Hence, a combination of (4.2), (4.3), (4.4) and $\Lambda^2(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2})=\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}=R(2\pi_1)$ shows that

$$\begin{aligned}\Lambda^2(\mathfrak{m}^{\mathbb{C}}) &= \big(R(2\pi_1)\hat{\otimes}\,\mathrm{Sym}^2\,\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\oplus\big((R(4\pi_1)\oplus 1)\hat{\otimes}\Lambda^2\,\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\\ &= \big(R(2\pi_1)\hat{\otimes}R(2\pi_1+2\pi_{q-1})\big)\oplus\big(R(2\pi_1)\hat{\otimes}R(\pi_2+\pi_{q-2})\big)\oplus\big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\oplus R(2\pi_1)\\ &\quad\oplus\big(R(4\pi_1)\hat{\otimes}R(2\pi_1+\pi_{q-2})\big)\oplus\big(R(4\pi_1)\hat{\otimes}R(\pi_2+2\pi_{q-1})\big)\oplus\big(R(4\pi_1)\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\\ &\quad\oplus R(2\pi_1+\pi_{q-2})\oplus R(\pi_2+2\pi_{q-1})\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}.\end{aligned}$$

We deduce that $\mathfrak{m}^{\mathbb{C}}$ appears once inside $\Lambda^2(\mathfrak{m}^{\mathbb{C}})$ and since $\mathfrak{m}$ is of real type, it follows that $\mathbf{a}=1$.

Let us treat the decomposition of the second symmetric power. We start with the low dimensional case $p=2$, $q=3$, i.e. $\mathfrak{m}^{\mathbb{C}}=\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}$. A combination of (4.2), (4.3) and (4.4), yields

$$\begin{aligned}\mathrm{Sym}^2(\mathfrak{m}^{\mathbb{C}}) &= \Big(\big(R(4\pi_1)\oplus 1\big)\hat{\otimes}\big(R(2\pi_1+2\pi_2)\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}\oplus 1\big)\Big)\oplus\Big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\big(R(3\pi_1)\oplus R(3\pi_2)\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}\big)\Big)\\ &= \big(R(4\pi_1)\hat{\otimes}R(2\pi_1+2\pi_2)\big)\oplus\big(R(4\pi_1)\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}\big)\oplus R(4\pi_1)\oplus R(2\pi_1+2\pi_2)\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}\oplus 1\\ &\quad\oplus\big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}R(3\pi_1)\big)\oplus\big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}R(3\pi_2)\big)\oplus\big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}\big).\end{aligned}$$

Hence, there is a copy of $\mathfrak{m}^{\mathbb{C}}$ inside $\mathrm{Sym}^2(\mathfrak{m}^{\mathbb{C}})$ and as above we conclude that $\mathbf{s}=1$. In a similar way, for $p=2$ and $q\leq 4$, we get that

$$\begin{aligned}\mathrm{Sym}^2(\mathfrak{m}^{\mathbb{C}}) &= \big(R(4\pi_1)\oplus 1\big)\hat{\otimes}\big(R(2\pi_1+2\pi_{q-1})\oplus R(\pi_2+\pi_{q-2})\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\oplus 1\big)\\ &\quad\oplus\Big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\big(R(2\pi_1+\pi_{q-2})\oplus R(\pi_2+2\pi_{q-1})\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\Big)\\ &= \big(R(4\pi_1)\hat{\otimes}R(2\pi_1+2\pi_{q-1})\big)\oplus\big(R(4\pi_1)\hat{\otimes}R(\pi_2+\pi_{q-2})\big)\oplus\big(R(\pi_2+\pi_{q-2})\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\\ &\quad\oplus R(4\pi_1)\oplus R(2\pi_1+2\pi_{q-1})\oplus R(\pi_2+\pi_{q-2})\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\oplus 1\\ &\quad\oplus\big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}R(2\pi_1+\pi_{q-2})\big)\oplus\big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}R(\pi_2+2\pi_{q-1})\big)\oplus\big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big).\end{aligned}$$

Thus again we conclude $\mathbf{s}=1$.

Case B: $3\leq p\leq q$. In this case, a combination of (4.2), (4.3) and (4.4) yields the following decomposition for any $p\geq 3$ and $q\geq p$:

$$\begin{aligned}\Lambda^2(\mathfrak{m}^{\mathbb{C}}) &= \big(\Lambda^2R(\pi_1+\pi_{p-1})\hat{\otimes}\,\mathrm{Sym}^2\,R(\pi_1+\pi_{q-1})\big)\oplus\big(\mathrm{Sym}^2\,R(\pi_1+\pi_{p-1})\hat{\otimes}\Lambda^2R(\pi_1+\pi_{q-1})\big)\\ &= \big(R(2\pi_1+\pi_{p-2})\oplus R(\pi_2+2\pi_{p-1})\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\big)\hat{\otimes}\big(R(2\pi_1+2\pi_{q-1})\oplus R(\pi_2+\pi_{q-2})\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\oplus 1\big)\\ &\quad\oplus\big(R(2\pi_1+2\pi_{p-1})\oplus R(\pi_2+\pi_{p-1})\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\oplus 1\big)\hat{\otimes}\big(R(2\pi_1+\pi_{q-2})\oplus R(\pi_2+2\pi_{q-1})\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big).\end{aligned}$$

These two external tensor products each contain one copy of $\mathfrak{m}^{\mathbb{C}}$, hence we have $\mathbf{a}=2$ in this case. Passing to the second symmetric power and working in the same way we get that

$$\begin{aligned}\mathrm{Sym}^2(\mathfrak{m}^{\mathbb{C}}) &= \big(\mathrm{Sym}^2\,\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\hat{\otimes}\,\mathrm{Sym}^2\,\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\oplus\big(\Lambda^2\,\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\hat{\otimes}\Lambda^2\,\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\\ &= \big(R(2\pi_1+2\pi_{p-1})\oplus R(\pi_2+\pi_{p-2})\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\oplus 1\big)\hat{\otimes}\big(R(2\pi_1+2\pi_{q-1})\oplus R(\pi_2+\pi_{q-2})\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\oplus 1\big)\\ &\quad\oplus\big(R(2\pi_1+\pi_{p-2})\oplus R(\pi_2+2\pi_{p-1})\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\big)\hat{\otimes}\big(R(2\pi_1+\pi_{q-2})\oplus R(\pi_2+2\pi_{q-1})\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big).\end{aligned}$$

It follows that there are two instances of $\mathfrak{m}^{\mathbb{C}}$ inside $\mathrm{Sym}^2(\mathfrak{m}^{\mathbb{C}})$, i.e. $\mathbf{s}=2$.

To compute the dimension of the space of invariant three-forms, consider the additional equivariant isomorphism

$$\Lambda^3(V\otimes W)=\big(\Lambda^3V\otimes\mathrm{Sym}^3\,W\big)\oplus\big(P_V(2,1)\otimes P_W(2,1)\big)\oplus\big(\mathrm{Sym}^3\,V\otimes\Lambda^3W\big),$$

where $P_V(2,1)$ and $P_W(2,1)$ are the (2,1)-plethysms of V and W, respectively. From this we deduce that any invariant 3-form on $V\otimes W$ can be projected to component forms induced from the following cases:

- An invariant 3-form on V and an invariant symmetric 3-tensor on W
- The product of two invariant elements of the (2,1)-plethysms of V and of W
- An invariant symmetric 3-tensor on V and an invariant 3-form on W

Let us first compute such objects for $V := \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p} = R(\pi_1 + \pi_{p-1})$. We have

$$\Lambda^3 R(\pi_1 + \pi_{p-1}) \oplus P_{R(\pi_1+\pi_{p-1})}(2, 1) = \Lambda^2 R(\pi_1 + \pi_{p-1}) \otimes R(\pi_1 + \pi_{p-1}),$$

and by (4.3) we have $\Lambda^2 \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p} = \Lambda^2 R(\pi_1 + \pi_{p-1}) = R(\pi_1 + \pi_{p-1}) \oplus R(2\pi_1 + \pi_{p-2}) \oplus R(\pi_2 + 2\pi_{p-1})$. The first part of the computation is

$$\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p} \otimes \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p} = R(\pi_1 + \pi_{p-1}) \otimes R(\pi_1 + \pi_{p-1}) = R(2\pi_1 + 2\pi_{p-1}) \oplus R(\pi_1 + \pi_{p-1}) \oplus \mathbb{R},$$

where the trivial term $\mathbb{R}$ corresponds to the tensor of the Killing form of $\mathfrak{su}(p)$ (see Remark 3.16). This yields the first invariant 3-form on $R(\pi_1 + \pi_{p-1})$. Now we are left with the task of computing unsymmetrized tensor products of irreducible SU_p-modules, and this can be done via the Littlewood–Richardson rule. The result is that the products $R(2\pi_1+\pi_{p-2})\otimes R(\pi_1+\pi_{p-1})$ and $R(\pi_2 + 2\pi_{p-1}) \otimes R(\pi_1 + \pi_{p-1})$, do not contain the trivial representation. Hence the (2,1)-plethysm of $R(\pi_1 + \pi_{p-1})$ also does not admit any trivial modules. By (4.4) we also deduce that the multiplicity of V in $\mathrm{Sym}^2 R(\pi_1 + \pi_{p-1})$ is 0 for $p = 2$, or 1 for $p \geq 3$. Furthermore, we have

$$\mathrm{Sym}^3 R(\pi_1 + \pi_{p-1}) \oplus P_{R(\pi_1+\pi_{p-1})}(2, 1) = \mathrm{Sym}^2 R(\pi_1 + \pi_{p-1}) \otimes R(\pi_1 + \pi_{p-1})$$

and the tensor product between $R(\pi_1 + \pi_{p-1}) \subset \mathrm{Sym}^2 R(\pi_1 + \pi_{p-1})$ and the rightmost factor $R(\pi_1 + \pi_{p-1})$ contains a trivial submodule, corresponding as before to the tensor of the Killing form. This trivial submodule is contained in $\mathrm{Sym}^3 R(\pi_1 + \pi_{p-1})$, since we have already shown that $P_{R(\pi_1+\pi_{p-1})}(2, 1)$ contains no such. Note that the computations for $W := \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q} = R(\pi_1 + \pi_{q-1})$ are identical (except for switching the label p to q). Therefore, we finally get a one dimensional trivial submodule in $\Lambda^3(V \otimes W)$ for the case $p = 2$ (or q=2), and when $p, q \geq 3$ we get a two dimensional trivial submodule in $\Lambda^3(V \otimes W)$. In our notation, this means $\ell = 1$ for $p = 2, q \geq 3$ and $\ell = 2$ for $p, q \geq 3$. This completes the proof. □

Lemma 4.10. *Consider the homogeneous space $M = G/K = \mathrm{Sp}_n/(\mathrm{SO}_n \times \mathrm{Sp}_1)$ with $n \geq 3$. Then, the isotropy representation $\mathfrak{m} = R(2\pi_1)\hat{\otimes}\mathfrak{sp}(1)$ has multiplicity $\mathbf{a} = 1$ inside $\Lambda^2\mathfrak{m}$ and multiplicity $\mathbf{s} = 0$ inside $\mathrm{Sym}^2 \mathfrak{m}$. Moreover, the dimension of trivial submodule $(\Lambda^3\mathfrak{m})^K$ is $\ell = 1$.*

Proof. An embedding of a compact Lie group K into Sp_n is equivalent to a (faithful) representation $\phi : K \to \mathrm{GL}(\mathbb{H}^n)$. This is a representation of real dimension $4n$ with an invariant quaternionic structure. Since $K = \mathrm{SO}_n \times \mathrm{Sp}_1$ is compact, the image of ϕ will be inside some conjugacy class of Sp_n. We are looking for the unique isotropy irreducible embedding, which means that ϕ should be an irreducible representation. Let $R(\omega_1, \omega_2) = R(\omega_1)_{\mathrm{SO}_n} \hat{\otimes}_{\mathbb{C}} R(\omega_2)_{\mathrm{Sp}_1}$ denotes the associated real irreducible representation. The obvious candidate is $\phi = R(\pi_1, \pi_1) = \mathbb{R}^n \hat{\otimes}_{\mathbb{R}} \mathbb{H} = \mathbb{R}^n \hat{\otimes}_{\mathbb{R}} \mathbb{C}^2$. This irreducible representation is obviously of quaternionic type. Recall now the adjoint representation of Sp_n is the real submodule inside $\mathrm{Ad}^{\mathbb{C}}_{\mathrm{Sp}_n} = \mathrm{Sym}^2 \nu_n = \mathrm{Sym}^2_{\mathbb{C}} \mathbb{H}^n$. Thus we must take into account the complex structure on ϕ, which is defined by its action on the right tensor factor $\mathbb{H} \simeq \mathbb{C}^2$. By applying (4.2), we compute

$$\begin{aligned}\mathrm{Sym}^2_{\mathbb{C}} \phi &= \mathrm{Sym}^2_{\mathbb{C}}(\mathbb{R}^n \hat{\otimes}_{\mathbb{R}} \mathbb{C}^2) = (\mathrm{Sym}^2_{\mathbb{R}} \mathbb{R}^n \hat{\otimes}_{\mathbb{R}} \mathrm{Sym}^2_{\mathbb{C}} \mathbb{C}^2) \oplus (\Lambda^2_{\mathbb{R}} \mathbb{R}^n \hat{\otimes} \Lambda^2_{\mathbb{C}} \mathbb{C}^2) \\ &= (R(2\pi_1) \oplus R(0)) \otimes \mathrm{Ad}^{\mathbb{C}}_{\mathrm{Sp}_1} \oplus \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SO}_n}.\end{aligned}$$

This immediately yields the isotropy representation

$$\mathfrak{m} = \mathfrak{sp}(n)/(\mathfrak{so}(n) \oplus \mathfrak{sp}(1)) = R(2\pi_1)\hat{\otimes}\mathfrak{sp}(1) = R(2\pi_1, 2\pi_1),$$

which is irreducible. Since $\mathfrak{m}$ has real type and its tensor factors also have real type, we can apply complex representation theory without performing any extra complexifications. We proceed with the decomposition of $\Lambda^2\mathfrak{m}$ and $\mathrm{Sym}^2 \mathfrak{m}$. For any $n > 4$ note the following decompositions of SO_n-modules: $\Lambda^2 R(2\pi_1) = R(2\pi_1 + \pi_2) \oplus R(\pi_2)$ and $\mathrm{Sym}^2 R(2\pi_1) = R(4\pi_1) \oplus R(2\pi_1) \oplus R(2\pi_2) \oplus R(0)$. For Sp_1 we have that $\Lambda^2\mathfrak{sp}(1) = \mathfrak{sp}(1)$ and $\mathrm{Sym}^2 \mathfrak{sp}(1) = R(4\pi_1) \oplus R(0) = R(4\pi_1) \oplus 1$. Hence we conclude that only those terms in the tensor square that contain a factor of $\mathrm{Sym}^2 R(2\pi_1)$ and $\Lambda^2\mathfrak{sp}(1)$ will yield copies of $\mathfrak{m}$. In particular, the decomposition

$$\Lambda^2\mathfrak{m} = \Lambda^2\big(R(2\pi_1)\hat{\otimes}\mathfrak{sp}(1)\big) = \big(\Lambda^2 R(2\pi_1)\hat{\otimes}\, \mathrm{Sym}^2 \mathfrak{sp}(1)\big) \oplus \big(\mathrm{Sym}^2 R(2\pi_1)\hat{\otimes}\Lambda^2\mathfrak{sp}(1)\big)$$

contains precisely one instance of $\mathfrak{m}$, i.e. $\mathbf{a} = 1$. One the other hand,

$$\mathrm{Sym}^2 \mathfrak{m} = \mathrm{Sym}^2\big(R(2\pi_1)\hat{\otimes}\mathfrak{sp}(1)\big) = \big(\Lambda^2 R(2\pi_1)\hat{\otimes}\Lambda^2\mathfrak{sp}(1)\big) \oplus \big(\mathrm{Sym}^2 R(2\pi_1)\hat{\otimes}\, \mathrm{Sym}^2 \mathfrak{sp}(1)\big),$$

hence $\mathbf{s} = 0$. This proves the claim for $n > 4$. For completeness we examine the low-dimensional cases. Let first $n = 3$. The defining representation ϕ of $K = \mathrm{SO}_3 \times \mathrm{Sp}_1$ must be of real dimension $12 = 3 \times 4$ and hence the only irreducible possibility is $\phi = \mathbb{R}^3\hat{\otimes}\mathbb{H} = R(2\pi_1)\hat{\otimes}R(\pi_1)$. Thus we get

$$\mathrm{Sym}^2_{\mathbb{C}} \phi = \mathrm{Sym}^2_{\mathbb{C}}(\mathbb{R}^3 \hat{\otimes}_{\mathbb{R}} \mathbb{C}^2) = \big(R(4\pi_1)\hat{\otimes}\mathfrak{sp}(1)\big)^{\mathbb{C}} \oplus \big(\mathfrak{sp}(1) \oplus \mathfrak{so}(3)\big)^{\mathbb{C}}.$$

Hence in this case $\mathfrak{m} = R(4\pi_1)\hat{\otimes}\mathfrak{sp}(1)$. As $\mathfrak{so}(3)$-modules, we have that

$$\Lambda^2 R(4\pi_1) = R(6\pi_1) \oplus \mathfrak{so}(3), \quad \mathrm{Sym}^2 R(4\pi_1) = R(8\pi_1) \oplus R(4\pi_1) \oplus R(0).$$

Therefore, only products of $\mathrm{Sym}^2 R(4\pi_1)$ and $\Lambda^2\mathfrak{sp}(1)$ yield copies of $\mathfrak{m}$. Consequently, the result is the same as above, the multiplicity of $\mathfrak{m}$ is one in $\Lambda^2\mathfrak{m}$ and zero in $\mathrm{Sym}^2\,\mathfrak{m}$.

Assume now that $n = 4$. The defining representation of $K = \mathrm{SO}_4 \times \mathrm{Sp}_1$ is $\phi = \mathbb{R}^4\hat{\otimes}\mathbb{H}$, but $\mathbb{R}^4 = R(\pi_1 + \pi_2)$ in terms of highest weights, instead of being $R(\pi_1)$ as before, because SO_4 is non-simple. We get

$$\mathrm{Sym}^2_{\mathbb{C}}\,\phi = \mathrm{Sym}^2_{\mathbb{C}}(\mathbb{R}^4\hat{\otimes}_{\mathbb{R}}\mathbb{C}^2) = \big(R(2\pi_1 + 2\pi_2)\hat{\otimes}\mathfrak{sp}(1)\big)^{\mathbb{C}} \oplus \big(\mathfrak{sp}(1) \oplus \mathfrak{so}(4)\big)^{\mathbb{C}}$$

and thus $\mathfrak{m} = R(2\pi_1 + 2\pi_2)\hat{\otimes}\mathfrak{sp}(1)$ in this case. As $\mathfrak{so}(4)$-modules, we see that

$$\begin{aligned}\Lambda^2 R(2\pi_1 + 2\pi_2) &= R(2\pi_1 + 4\pi_2) \oplus R(4\pi_1 + 2\pi_2) \oplus \mathfrak{so}(3),\\ \mathrm{Sym}^2 R(2\pi_1 + 2\pi_2) &= R(4\pi_1 + 4\pi_2) \oplus R(4\pi_1) \oplus R(4\pi_2) \oplus R(2\pi_1 + 2\pi_2) \oplus R(0),\end{aligned}$$

and the same argument as previously yields that $\mathbf{a} = 1$ and $\mathbf{s} = 0$.

Now, our assertion for $(\Lambda^3\mathfrak{m})^K$ can be deduced very easily as follows: Any invariant element of $\Lambda^3\mathfrak{m}$ induces an equivariant map in $\mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m})$. For any $n \geq 3$ we have shown that $\mathbf{a} = \dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m}) = 1$. Thus, $\dim_{\mathbb{R}}(\Lambda^3\mathfrak{m})^K \leq 1$, but by Lemma 3.12 we also get $\dim_{\mathbb{R}}(\Lambda^3\mathfrak{m})^K \geq 1$ and the result follows. Note that this method for the computation of the multiplicity ℓ, applies on any non-symmetric SII space $M = G/K$ whose isotropy representation is of real type and has $\mathbf{a} = 1$ (or whose isotropy representation is of complex type and has $\mathbf{a} = 2$, e.g. $\mathrm{S}^6 \cong \mathrm{G}_2/\mathrm{SU}_3$). $\square$

5. Classification of homogeneous ∇-Einstein structures on SII spaces

5.1. Homogeneous ∇-Einstein structures

Similarly with invariant Einstein metrics on homogeneous Riemannian manifolds, on triples (M^n, g, T) consisting of a homogeneous Riemannian manifold $(M^n = G/K, g)$ endowed with a (non-trivial) invariant 3-form T, on may speak of *homogeneous ∇-Einstein structures*. In particular,

Definition 5.1. A triple (M^n, g, T) of a connected Riemannian manifold (M, g) carrying a (non-trivial) 3-form $T \in \Lambda^3 T^*M$, is called a *G-homogeneous ∇-Einstein manifold* (with skew-torsion) if there is a closed subgroup $G \subseteq \mathrm{Iso}(M, g)$ of the isometry group of (M, g), which acts transitively on M and a G-invariant connection ∇ compatible with g and with skew-torsion T, whose Ricci tensor satisfies the condition (1.2).

In this case, g is a G-invariant metric, the Levi-Civita connection ∇^g is a G-invariant metric connection and since $2(\nabla - \nabla^g) = T$, the torsion T of ∇ is given necessarily by a G-invariant 3-form $0 \neq T \in \Lambda^3(\mathfrak{m})^K$, where $\mathfrak{m} \cong T_oM$ is a reductive complement of $M = G/K$ with $K \subset G$ being the (closed) isotropy group. In particular, the ∇-Einstein condition (1.2) is $\mathrm{Ad}(K)$-invariant, in the sense that the Ricci tensor Ric^∇ is a G-invariant covariant 2-tensor which is described by an $\mathrm{Ad}(K)$-invariant bilinear form on $\mathfrak{m}$, and the same for its symmetric part. Moreover,

Proposition 5.2. *On a homogeneous Riemannian manifold $(M = G/K, g)$ carrying a G-invariant (non-trivial) 3-form $T \in \Lambda^3(\mathfrak{m})^K$, the scalar curvature $\mathrm{Scal} = \mathrm{Scal}^\nabla$ associated to the G-invariant metric connection $\nabla := \nabla^g + \frac{1}{2}T$ is a constant function on M.*

Proof. It is well-known that on a reductive homogeneous space, the scalar curvature Scal^g of the Levi-Civita connection (related to a G-invariant Riemannian metric g, or the corresponding $\mathrm{Ad}(K)$-invariant inner product $\langle\ ,\ \rangle$ on the reductive complement $\mathfrak{m}$) is independent of the point, i.e. it is a constant function on M [13,35]. Let ∇ be a G-invariant metric connection on $(M = G/K, g)$ whose skew-torsion coincides with the invariant 3-form $0 \neq T \in \Lambda^3(\mathfrak{m})^K$. Due to the identity $\mathrm{Scal} = \mathrm{Scal}^g - \frac{3}{2}\|T\|^2$ it is sufficient to prove that $\|T\|^2$ is constant, which is obvious since T corresponds to a G-invariant tensor field. Consequently, $\mathrm{Scal}^\nabla : G/K \to \mathbb{R}$ is constant. $\square$

5.2. On the proofs of Theorems C–E

Let us focus now on an effective, non-symmetric (compact) strongly isotropy irreducible homogeneous Riemannian manifold $(M = G/K, g = -B|_{\mathfrak{m}})$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the associated B-orthogonal decomposition. We denote by $\mathcal{M}^{sk}_G(\mathrm{SO}(G/K, g)) \subseteq \mathcal{M}_G(\mathrm{SO}(G/K, g)) \subseteq \mathcal{A}\mathit{ff}_G(F(G/K))$ the space of G-invariant affine connections on $M = G/K$ which are compatible with the Killing metric $g = -B|_{\mathfrak{m}}$ and have invariant 3-forms $0 \neq T \in (\Lambda^3\mathfrak{m})^K$ as their torsion tensors. For the corresponding set of homogeneous ∇-Einstein structures, we will write $\mathcal{E}^{sk}_G(\mathrm{SO}(G/K, g))$. As stated in the introduction, Lemma 3.12 and Schur's lemma allow us to parametrize $\mathcal{E}^{sk}_G(\mathrm{SO}(G/K, g))$ by the space of global G-invariant 3-forms. Hence, this finally yields the identification $\mathcal{E}^{sk}_G(\mathrm{SO}(G/K, g)) = \mathcal{M}^{sk}_G(\mathrm{SO}(G/K, g))$. With the aim to clarify this identification and give explicit proofs of Theorems C–E in introduction, let us recall first the following important result of [17].

Theorem 5.3 ([17, Thm. 4.7]). *Let $(M^n = G/K, g)$ be an effective, compact and simply-connected, isotropy irreducible standard homogeneous Riemannian manifold $(M^n = G/K, g)$ of a compact connected simple Lie group G, which is not a symmetric space of Type I. Then, $(M^n = G/K, g)$ is a ∇^α-Einstein manifold for any parameter $\alpha \neq 0$, where $\nabla^\alpha = \nabla^g + \frac{1}{2}T^\alpha = \nabla^c + \Lambda^\alpha$ is the 1-parameter family of G-invariant metric connections on M, with skew-torsion $0 \neq T^\alpha = \alpha \cdot T^c$ (see Lemma 3.4).*

Note that for a symmetric space $M = G/K$ of Type I, the associated space of G-invariant affine *metric* connections is always a point, i.e. $\nabla^\alpha \equiv \nabla^c \equiv \nabla^g$ and no torsion appears. On the other hand, Theorem 5.3 generalizes the well-known fact that a compact simple Lie group G is ∇^α-Einstein with (non-trivial) *parallel torsion* for any $0 \neq \alpha \in \mathbb{R}$, with the flat ± 1-connections of Cartan–Schouten being the trivial members (see for example [3, Lemma 1.8] or [17, Thm. 1.1]). Notice however, that if $M = G/K$ is not isometric to a compact simple Lie group, then the ∇^α-Einstein structures described in Theorem 5.3 have parallel torsion only for $\alpha = 1$. We finally remark that both $\mathrm{S}^6 = \mathrm{G}_2/\mathrm{SU}_3$, $\mathrm{S}^7 = \mathrm{G}_2/\mathrm{Spin}_7$ are (strongly) isotropy irreducible and non-symmetric, hence they are ∇^α-Einstein manifolds with skew-torsion, for any $0 \neq \alpha \in \mathbb{C}$, $0 \neq \alpha \in \mathbb{R}$, respectively (due to the type of their isotropy representation). The same applies for any compact, non-symmetric, effective SII homogeneous Riemannian manifold and this gives rise to Theorem C which is an immediate consequence of Theorem A.1 and Theorem 5.3.

Let us proceed now with a proof of Theorems D and E.

Proof of Theorem D. If $M = G/K$ is a manifold in Table 2 whose isotropy representation is of real type, different than $\mathrm{SO}_{10}/\mathrm{Sp}_2$, then Theorem A.2(i) ensures the existence of a second (real) 1-parameter family of G-invariant connections $\nabla^t \neq \nabla^a$, compatible with the Killing metric and with skew-torsion T^t such that $T^t \neq T^c \sim T^\alpha$ (for any $t, \alpha \in \mathbb{R}$), where T^c is the torsion of the (unique) canonical connection corresponding to $\mathfrak{m}$. Thus, we can write $\nabla^t = \nabla^g + \frac{1}{2}T^t$ with $\nabla^t \neq \nabla^\alpha$. Since $\mathrm{Ric}^{\nabla^t} \equiv \mathrm{Ric}^t$ is G-invariant, the same is true for $\delta^g T^t = \delta^{\nabla^t} T^t$, in particular we can view the codifferential of $T^t \in (\Lambda^3\mathfrak{m})^K$ as a G-invariant 2-form. However, χ is of real type, hence the trivial representation $\mathbb{R}$ does not appear in $\Lambda^2\mathfrak{m}$, i.e. $(\Lambda^2\mathfrak{m})^K = 0$. Hence, we deduce that $\delta^g T^t = 0 = \delta^t T^t$ and since $\mathfrak{m}$ is (strongly) isotropy irreducible over $\mathbb{R}$ and the Ricci tensor Ric^t is symmetric, by Schur's lemma it must be a multiple of the Killing metric, i.e. $(M = G/K, -B|_{\mathfrak{m}}, \nabla^t)$ is ∇^t-Einstein with skew-torsion. Our final claim follows now in combination with Theorem 5.3. □

It is well-known that an effective SII homogeneous space $M = G/K$ admits an (integrable) G-invariant complex structure if and only if is a Hermitian symmetric space [46]. Moreover, the existence of an invariant *almost* complex structure $J \in \mathrm{End}(\mathfrak{m})$ on a strongly isotropy irreducible space implies that the isotropy representation is not of real type, hence $\chi = \phi \oplus \overline{\phi}$ for some irreducible complex representation with $\phi \ncong \overline{\phi}$. Consequently, any manifold which appears in Tables 4, 5 and whose isotropy representation is of complex type, is a G-homogeneous almost complex manifold (see also [46, Cor. 13.2]). Notice also

Lemma 5.4. *Let $\mathfrak{k}$ be a compact Lie algebra and let $\rho : \mathfrak{k} \to \mathrm{End}(\mathfrak{m})$ be a faithful (irreducible) representation of $\mathfrak{k}$ over $\mathbb{R}$, endowed with an invariant inner product $B_{\mathfrak{m}}$. Assume that $\dim \mathfrak{m} \geq 2$. If $\mathfrak{m}$ admits an $\mathrm{ad}(\mathfrak{k})$-invariant complex structure J (as a vector space), then $\Lambda^2\mathfrak{m}$ contains the trivial representation $\mathbb{R}$.*

Proof. We only mention that since $\mathfrak{m}$ is an irreducible complex type representation of a compact Lie algebra $\mathfrak{k}$, it is unitary, therefore the $\mathrm{ad}(\mathfrak{k})$-invariant Kähler form $\omega(X, Y) = B_{\mathfrak{m}}(JX, Y)$ gives rise to an $\mathrm{ad}(\mathfrak{k})$-invariant element inside $\Lambda^2\mathfrak{m}$. □

Consider now the spaces $\mathrm{SO}_{n^2-1}/\mathrm{SU}_n$ $(n \geq 4)$ and $\mathrm{E}_6/\mathrm{SU}_3$. Since their isotropy representation is of complex type, Lemma 5.4 certifies the existence of G-invariant 2-forms. Thus, in contrast to Theorem D, we cannot deduce that the Ricci tensor of all predicted G-invariant metric connections with skew-torsion must be necessarily symmetric (although this is the case always for Ric^α). However, since we are considering the isotropy irreducible case, we obtain Theorem E as follows:

Proof of Theorem E. Assume that $(M = G/K, g = -B|_{\mathfrak{m}})$ is one of the manifolds $\mathrm{SO}_{n^2-1}/\mathrm{SU}_n$ $(n \geq 4)$ or $\mathrm{E}_6/\mathrm{SU}_3$. By Theorem A.2(ii) (see also Table 2) we know that $M = G/K$ admits a 4-dimensional space of G-invariant metric connections with skew-torsion. Now, the ∇-Einstein condition is related only with the symmetric part of the Ricci tensor associated to any such connection. Since this tensor is G-invariant, Schur's lemma ensures that the ∇-Einstein equation is satisfied for any available G-invariant metric connection ∇ with skew-torsion. Therefore, the space of G-invariant ∇-Einstein structures has the same dimension with the space of G-invariant metric connections with skew-torsion. This proves Theorem E. □

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