M U N I

Annex No. 10 to the MU Directive on Habilitation Procedures and Professor Appointment Procedures

HABILITATION THESIS REVIEWER'S REPORT

Masaryk University

Applicant	John Bourke
Habilitation thesis	Categorical structures for higher-dimensional universal algebra
Reviewer	Nicola Gambino, Associate Professor in Pure Mathematics and Director of Research and Innovation
Reviewer's home unit, institution	School of Mathematics, University of Leeds, Woodhouse Lane, Leeds LS2 9JT, United Kingdom

Review

It is with pleasure that I provide this review for John Bourke's habilitation thesis "Categorical structures in higher-dimensional algebra".

Bourke's research focuses on category theory, a relatively young branch of pure mathematics that arose from the work of Mac Lane and Eilenberg in algebraic topology in the mid 1940s. Since then, the discipline has grown significantly, both to provide a convenient language to describe universal properties in a variety of contexts (most notably in algebraic geometry and algebraic topology) and as an independent area, with its own internal subdisciplines, open problems and research directions. Bourke's work focuses on what may be called higher-dimensional category theory, which studies variants of the notion of a category in which one has not only 0-cells (objects) and 1-cells (morphisms) between them, but also n-cells between (n-1)-cells for either some or all values of $n > 1$. For example, there is a 2-category of monoidal categories, strong monoidal functors and monoidal natural transformations. This notoriously challenging research area has significant applications to both within category theory itself, to algebraic topology, as well as mathematical logic and theoretical computer science. Bourke was trained in this area by the Australian school and has since developed into a very bright researcher in the area, strengthening the research group at Masaryk University led by Jiri Rosicky.

The contents of Bourke's thesis provide ample evidence of his ability, hard work and creativity as a mathematician: of the 5 papers in the thesis, two are published in the highly prestigious journal *Advances in Mathematics* and two in the well-regarded specialist journals *Journal of Pure and Applied Algebra* and *Journal of Homotopy and Related Structures.*

The contributions in these papers are important, original, technically very challenging and cover a very broad range of topics. I would like to bring attention to this last point, as I feel it is important to evaluate the future potential of a researcher. Bourke's work covers several separate strands of research in higher-dimensional category theory: 2-dimensional algebra, the theory of algebraic weak factorisation systems, monoidal categories, algebraic notions of weak higher categories, and abstract homotopy theory. By having made inroads into each topic, he has ample choice of future directions to pursue, which bodes well for his career.

As mentioned above, the papers included in the thesis make several important contributions. I would like to discuss only two, so as to give the flavour of Bourke's ability. First, in his paper "Two-dimensional monadicity", Bourke proves a theorem which characterises what we may call 2-categories of algebraic structures (like that of monoidal categories), just as Beck's fundamental result in the 60's characterised 1-dimensional categories of algebraic structures (like groups). Bourke's result is not only more challenging to prove, but its proof required the use of ideas and techniques that were neither part of Beck's work nor considered immediately relevant to the problem at hand. Bourke's result is clean and memorable: it provides necessary and sufficient conditions, easily verifiable in examples, that allow us to recognize when a 2-category is the category of algebras for a 2-monad (such as the 2-category of monoidal categories). In short, it has all the hallmarks of a classical result.

The second result that I would like to highlight is from his paper "Monads and Theories". The work therein is the culmination of a long and well-established line of research that began in the '60s with the characterisation by Linton of finitary functors as those induced by Lawvere theories. In recent years, several authors considered extensions of Lawvere theories in order to accommodate the more complex notions of signature needed to describe composition of n-cells in a higher-dimensional category. Yet, none of the notions of theory introduced was sufficiently general to include the example supporting the globular definition of higher groupoids (important in view of Grothendieck's homotopy hypothesis). Bourke identified a class of theories capturing this example and characterised completely the monads associated to them. Importantly, the characterisation involves a notion that was already present in the literature and that is of independent interest. Again, he proved a clean and beautiful result with an important application.

For brevity, I will not keep going. But please rest assured that the thesis contains several other results of similar calibre: the extension of the theory of algebraic weak factorisation systems to allow for double categories of generators and the results on skew monoidal categories are other highlights. I should also mention that all these results have already proved to be useful to other researchers, in algebraic topology, mathematical logic and category theory itself, and provide promising research directions for the future.

There is a pattern in these results: when faced with a difficult problem, Bourke succeeds in extending the existing theory to solve the problem in an elegant and natural way. Like the best research in category theory, his work makes hard problems look, after the fact, as if they were much easier to solve than they originally were. I look forward to his future work.

Reviewer's questions for the habilitation thesis defence

1. In which area outside category theory, if any, do you see your research having most impact or influence?

Conclusion

The habilitation thesis entitled "Categorical structures for higher-dimensional universal algebra" by John Bourke **fulfils** the requirements expected of a habilitation thesis in the field of Mathematics – Algebra and Theory of Numbers.

Date: May 1st, 2021 Signature: